London Theory Institute Lectures Series

Chris Hull
“ Duality Symmetry in String Theory ”

Informations

Pre-recorded Lectures : Youtube

Live Tutorial : Monday 22nd of November, 10h30

Abstract

This lecture provides an introduction to duality symmetries in string theory.

String theory was originally formulated as a theory of strings propagating in space time with interactions governed by the string coupling constant g. Scattering amplitudes for small g were constructed as a perturbation theory in g. Five consistent supersymmetric string theories were found, all in 10 spacetime dimensions with five distinct perturbation theories. This left many questions unanswered, such as why there should be five apparently consistent quantum theories of gravity and what happens to these theories as the coupling constant is increased.

Such questions were answered by the developments in the mid-1990s that have been called the 2nd superstring revolution. Dualities proved to be the key to uncovering the non-perturbative structure of superstring theory and in particular its strong coupling behaviour. When g is large, one can analyse the theory as a perturbation theory in 1/g and seek a “dual theory” with coupling constant g' whose perturbative expansion in g' matches the behaviour of the original theory as a perturbation theory in 1/g on identifying g'=1/g. In some cases the dual theory is again a string theory, which might be a different string theory from the original one. In other cases, the dual theory isn't a string theory at all, but a new theory – M-theory. This leads to a picture in which all 5 string theories are related by dualities and so are all seen as different limits of M-theory. Duality transformation provide new symmetries of string/M theory and T,S and U-dualities. Remarkably, the theory that emerges Is no longer just a theory of strings but one which includes both strings and branes which are higher dimensional extended objects. As the branes are related to strings by duality symmetries, they should be regarded as being on the same footing as the strings and of equal importance.

The lecture explores all of these issues and discusses some examples.

Duality Symmetries in String Theory: Problems

1) T-duality.

a) Consider the bosonic string propagating on a spacetime $M_D \times S^1$ which is a product of D-dimensional Minkowski space with coordinates X^μ (with $D = 25$) and a circle of radius R with coordinate Y which is periodic $Y \sim Y + 2\pi$. The action is

$$S = \frac{1}{2}T \int d^2\sigma \sqrt{h} \left[\partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + R^2 \partial_a Y \partial^a Y\right] \tag{1}$$

Here $\sigma^a = ('\sigma^1, \sigma^2)$ are the world-sheet coordinates and h_{ab} is the world-sheet metric, with $h = |\det h_{ab}|$. This can be coupled to a world-sheet gauge field A_a by the minimal coupling

$$\partial_a Y \to D_a Y \equiv \partial_a Y + A_a$$

so that the action becomes

$$S = \frac{1}{2}T \int d^2\sigma \sqrt{h} \left[\partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + R^2 D_a Y D^a Y\right]$$

What is the gauge symmetry of this action for which A_a is the gauge field? What theory results from integrating out A?

b) Further modify the action by adding a gauge-invariant term $\tilde{Y} F$ where $\tilde{Y}(\sigma)$ is a new world-sheet field and F is the field strength

$$F_{ab} = \partial_a A_b - \partial_b A_a$$

so that the action becomes

$$S = \frac{1}{2}T \int d^2\sigma \sqrt{h} \left[\partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + R^2 D_a Y D^a Y\right] + \alpha \frac{c}{2} \int d^2\sigma \sqrt{h}\, \tilde{Y} \epsilon^{ab} F_{ab} \tag{2}$$

where ϵ^{ab} is the alternating tensor on the world-sheet with $\epsilon^{12} = -\epsilon^{21} = h^{-1/2}$, c is a real constant that we will determine later and $\alpha = 1$ if the world-sheet metric h_{ab} has Lorentzian signature and $\alpha = i$ if it has Euclidean signature. (On analytically continuing from Lorentzian to Euclidean signature, ϵ^{ab} is usually taken to go to $i\epsilon^{ab}$.) The tensor satisfies

$$\epsilon^{ab}\epsilon_{bc} = \alpha^2 \delta^a{}_c$$

(where $\alpha^2 = 1$ in Lorentzian signature and $\alpha^2 = -1$ in Euclidean signature). Assume for now that the topology of the world-sheet is such that there are no winding modes and that all flat gauge fields are pure gauge, i.e. if $F = 0$, then A must be of the form $A_a = \partial_a \lambda$ for some λ. Show that on integrating out $\tilde{Y}$ and making a gauge choice, the original string action (1) is recovered. Alternatively, an action for $X, \tilde{Y}$ can be obtained by imposing a gauge condition on Y and integrating out A_μ. Show that this action describes a string moving on a product of M_D and a circle of radius $\tilde{R}$ where $\tilde{R} = c/TR$.

c) We now drop the restriction on the world-sheet topology and consider the action

$$S = \frac{1}{2} T \int d^2\sigma \sqrt{h} \left[\partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + R^2 D_a Y D^a Y \right] - \alpha c \int d^2\sigma \sqrt{h} \epsilon^{ab} \partial_a \tilde{Y} A_b$$

Check that this differs from (2) by a total derivative term. Consider the case that the world-sheet is a 2-torus, so that

$$\sigma^a \sim \sigma + 2\pi$$

with Euclidean signature, so that $\alpha = i$. If $\tilde{Y}$ is taken to be periodically identified, $\tilde{Y} \sim \tilde{Y} + 2\pi$, then $\tilde{Y}$ can have winding modes so that

$$\tilde{Y} = w_a \sigma^a + f(\sigma^a)$$

where f is a function periodic in both σ^1 and σ^2. Show that the winding numbers w_1, w_2 must be integers. Similarly, the gauge parameter λ appearing in the transformation

$$A_a \to A_a + \partial_a \lambda$$

can have winding modes, so that

$$\lambda = n_a \sigma^a + F(\sigma)$$

with integral winding numbers n_1, n_2 and doubly periodic function F. Show that the action is invariant up to total derivative terms under transformations with $n_a = 0$. For transformations that have $n_a \neq 0$, find the value of the constant c that ensure that the variation of the action under any such transformation is $2\pi i$ times an integer. Hence deduce that the quantum theory will be invariant under gauge transformations with winding number for this choice of c.

d) On a toroidal world-sheet, a flat gauge field with $F = 0$ is a constant gauge field plus a pure gauge piece, so that

$$A_a = \bar{A}_a + \partial_a \lambda$$

for some constant vector $\bar{A}_a$. Check that the action now contains a constant term

$$S = -c(2\pi)^2 \epsilon^{ab} w_a \bar{A}_b$$

Now integrating out $\tilde{Y}$ will include a sum over winding modes w_a while integrating out A_a will include an integral over $\bar{A}_b$. Show that this then implies the equivalence between strings on the product of M_D with a circle of radius R and strings on the product of M_D with a circle of radius $\tilde{R}$ where $\tilde{R} = c/TR$.

2) B-Shifts

a) The string world-sheet action for the bosonic string propagating in a background with metric $g_{\mu\nu}$, B-field $B_{\mu\nu}$ and dilaton Φ is

$$S = \int d^2\sigma \sqrt{h} \left[\frac{1}{2} T \left(\partial_a X^\mu \partial^a X^\nu g_{\mu\nu} + \alpha \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \right) + \frac{1}{4\pi} \Phi R(h) \right] \quad (3)$$

(with notation as in question 1). Show that the action is invariant under transformations of the form

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu} \Lambda_{\nu]}$$

while the field equation for X^μ is invariant under the more general transformations

$$B_{\mu\nu} \to B_{\mu\nu} + \omega_{\mu\nu}$$

provided

$$\partial_{[\mu} \omega_{\nu\rho]} = 0$$

b) Consider the case in which the target space is a torus T^n so that the coordinates are periodic

$$X^\mu \sim X^\mu + 2\pi$$

and the world-sheet is a 2-torus, so that

$$\sigma^a \sim \sigma + 2\pi$$

with Euclidean signature, so that $\alpha = i$. Then X^μ can have winding modes so that

$$X^\mu = w^\mu_a \sigma^a + f^\mu(\sigma^a)$$

Consider transformations of the form

$$B_{\mu\nu} \to B_{\mu\nu} + \omega_{\mu\nu}$$

with *constant* $\omega_{\mu\nu}$. Show that the action changes by $2\pi i$ times an integer so that the quantum theory is invariant provided

$$\omega_{\mu\nu} = \frac{1}{2\pi} N + \mu\nu$$

where $N_{\mu\nu}$ has integer components.

3) $\mathbf{SL(2,\mathbb{Z})}$
a) $SL(2,\mathbb{R})$ is the group of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

whose entries are real numbers and whose determinant is one. Show that the 2×2 matrices whose entries a, b, c, d are *integers* and whose determinant is one also form a group – this is the group $SL(2,\mathbb{Z})$.
b) Show that the matrices S, T given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

each generate a subgroup of $SL(2,\mathbb{Z})$ and find the form of these subgroups.

c) The electric charge q and magnetic charge p (normalised so that they are integers) transform under $SL(2,\mathbb{Z})$ S-duality as

$$\begin{pmatrix} q \\ p \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Show that an electrically charged state with $q = 1$, $p = 0$ transforms to a dyon with charges q, p that are *relatively prime*, i.e. which have no common factors (other than ± 1). Deduce that the degeneracy of dyonic states with charges p, q that are relatively prime is the same as the degeneracy of electrically charged states with $q = 1$, $p = 0$.

4.
a) For two metrics $g, \tilde{g}$ related by a Weyl rescaling

$$\tilde{g}_{\mu\nu} = e^{2f} g_{\mu\nu}$$

for some $f(x)$, the corresponding scalar curvatures $R, \tilde{R}$ are related by

$$\tilde{R} = e^{-2f} \left[R - 2(n-1)\nabla^2 f - (n-2)(n-1)(\nabla f)^2 \right]$$

in n dimensions. The world-sheet formulation leads to a spacetime effective action in n dimensions for the metric, B-field and dilaton of the form

$$S_N = \int d^n x \sqrt{-g} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]$$

where H is the field strength for $B_{\mu\nu}$ This is sometimes called the 'string frame' action. Find a Weyl rescaling that brings the gravitational kinetic term to the standard form

$$S = \int d^n x \sqrt{-g} R + \dots$$

and find the rest of this 'Einstein frame' action.

b) The $SO(32)$ heterotic string gives an effective low energy supergravity theory with bosonic fields $g_{\mu\nu}, B_{\mu\nu}, \Phi, A_\mu$. The string frame action for these bosonic fields is

$$S_{het} = S_N - \frac{1}{4} \int d^{10} x \sqrt{-g} e^{-2\Phi} Tr F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu}$ is the Yang-Mills field strength and the dimension is $n = 10$.
The Type I string also has gauge group $SO(32)$ and has an an effective low energy supergravity theory with the same spectrum of fields. For the type I

string, denote the bosonic fields $g'_{\mu\nu}, B'_{\mu\nu}, \Phi', A'_\mu$. The string frame action for these bosonic fields is

$$S_I = \int d^{10}x\sqrt{-g'}\left[e^{-2\Phi'}(R' + 4\partial_\mu\Phi'\partial^\mu\Phi') - \frac{1}{12}H'_{\mu\nu\rho}H'^{\mu\nu\rho} - \frac{1}{4}e^{-\Phi'}TrF'_{\mu\nu}F'^{\mu\nu}\right]$$

Show that these two actions are related by

$$\begin{aligned} \Phi' &= -\Phi \\ g'_{\mu\nu} &= e^{-\Phi}g_{\mu\nu} \\ H' &= H \\ F' &= F \end{aligned}$$

What is the significance of this relation between the two actions?

Duality Symmetries in String Theory: References

Here are some suggestions for further reading.

References [1-7] are review papers. If you are only going to look at one, I suggest [1]. Chapter 8 of the book [2] is the one that is most relevant but earlier chapters give some of the background material used. Reference [6] gives a good introduction to T-duality and [7] gives a more comprehensive review of T-duality. Reference [8] gives the details of the dimensional reduction of the theory with metric, b-field and dilaton on an n-torus and shows how the $O(n,n)$ symmetry arises. [8-11] are some of the original papers.

References

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