LonTI Tutorial on Superconformal Index

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1) The superconformal index for $\mathcal{N}=2$ theories was defined as:

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-2\beta\{\tilde{Q}_{1\dot{-}}, \tilde{S}^{1\dot{-}}\}} \rho^{\frac{1}{2}\delta_{1-}} \sigma^{\frac{1}{2}\delta_{1+}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}}$$

$\rightsquigarrow$ States contributing to the index should satisfy:

$\Delta = 2j_2 + 2R - r$ as can be seen for all states in the Table

For the rest:

$$\begin{cases} \delta_{1-} = \Delta - 2j_1 - 2R - r & \text{for } \rho \\ \delta_{1+} = \Delta + 2j_1 - 2R - r & \text{for } \sigma \\ \tilde{\delta}_{2\dot{+}} = \Delta + 2j_2 + 2R + r & \text{for } \tau \end{cases}$$

evaluate these $\forall$ letters

letter	index
ϕ	$\rho\sigma$
$\lambda_{1\pm}$	$-\sigma\tau, -\rho\tau$
$\bar{\lambda}_{1\dot{+}}$	$-\tau^2$
$\bar{F}_{\dot{+}\dot{+}}$	$\sigma\rho\tau^2$
$\partial_{-\dot{+}}\lambda_{1+} + \partial_{+\dot{+}}\lambda_{1-} = 0$	$\sigma\rho\tau^2$
q	τ
$\bar{\psi}_{\dot{+}}$	$-\sigma\rho\tau$
$\partial_{\pm\dot{+}}$	$\sigma\tau, \rho\tau$

For the vector we have: $\quad i_{vec} = \dfrac{\sigma\rho - \sigma\tau - \rho\tau - \tau^2 + 2\sigma\rho\tau^2}{(1-\sigma\tau)(1-\rho\tau)}$

Simplify to the following thing: $\quad i_{vec} = -\dfrac{\sigma\tau}{1-\sigma\tau} - \dfrac{\rho\tau}{1-\rho\tau} + \dfrac{\sigma\rho - \tau^2}{(1-\sigma\tau)(1-\rho\tau)}$

For the hyper (which is 2 half-hypers)

$$i_{hyper} = \frac{2(\tau - \sigma\rho\tau)}{(1-\sigma\tau)(1-\rho\tau)} = \frac{2\tau(1-\sigma\rho)}{(1-\sigma\tau)(1-\rho\tau)}$$

Now for the P.E.. In the case of the $\frac{1}{2}$ hyper

$$
\begin{aligned}
P.E.[i_{\frac{1}{2}H}] &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{\tau^n - \tau^n\sigma^n\rho^n}{(1-\sigma^n\tau^n)(1-\rho^n\tau^n)}\right] \\
&= \exp\left[\sum_{n=1}^{\infty}\sum_{m,k=0}^{\infty} \frac{1}{n}(\tau^n - \tau^n\sigma^n\rho^n)\cdot\sigma^{nm}\tau^{nm}\rho^{kn}\tau^{kn}\right] \\
&= \exp\left[\sum_{n=1}^{\infty}\sum_{k,m=0}^{\infty} \frac{1}{n}\left(\tau^{n(1+m+k)}\sigma^{nm}\rho^{kn} - \tau^{(1+m+k)n}\sigma^{n(m+1)}\rho^{n(k+1)}\right)\right] \\
&= \exp\left[-\sum_{m,k=0}^{\infty} \log(1-\tau^{1+m+k}\sigma^m\rho^k) + \sum_{m,k=0}^{\infty} \log(1-\tau^{1+m+k}\sigma^{m+1}\rho^{k+1})\right] \\
&= \prod_{k,m=0}^{\infty} \frac{1-\tau^{m+k+1}\sigma^{m+1}\rho^{k+1}}{1-\tau^{m+k+1}\sigma^m\rho^k} = \prod_{k,m=0}^{\infty} \frac{1-\tau^{-1}(\sigma\tau)^{m+1}(\rho\tau)^{k+1}}{1-\tau(\sigma\tau)^m(\rho\tau)^k}
\end{aligned}
$$

$$\equiv \Gamma(\tau; \sigma\tau, \rho\tau)$$ ↲ Elliptic Gamma function

$= \Gamma(\sigma\rho\tau; \sigma\tau, \rho\tau)$

& we have implicitly used that:

$|\sigma\tau| < 1$, $|\rho\tau| < 1$ for geometric series

$|\tau(\sigma\tau)^m(\rho\tau)^k| < 1$, $|\tau^{-1}(\sigma\tau)^{m+1}(\rho\tau)^{k+1}| < 1$ for log

↳ $|\tau| < 1$ ↳ $|\sigma\tau\rho| < 1$

For the vector we can do the same thing to get:

$$P.E.[i_{vec}] = \prod_{m,k=0}^{\infty} \frac{(1-\sigma^{m+1}\tau^{m+k+1}\rho^k)(1-\sigma^m\tau^{m+k+1}\rho^{k+1})(1-\tau^{2+k+m}\sigma^m\rho^k)}{(1-\sigma^{m+1}\rho^{k+1}\tau^{m+k})(1-\sigma^{m+1}\rho^{k+1}\tau^{2+k+m})^2}$$

$$= \prod_{m,k=0}^{\infty} \frac{(1-(\sigma\tau)^{m+1}(\rho\tau)^{k})(1-(\rho\tau)^{k+1}(\sigma\tau)^{m})\,(1-\tau^{2}(\sigma\tau)^{m}(\rho\tau)^{k})}{(1-\tau^{-2}(\sigma\tau)^{m+1}(\rho\tau)^{k+1})\,(1-(\sigma\tau)^{m+1}(\rho\tau)^{k+1})^{2}}$$

inverse elliptic Gamma function

$$= \prod_{m,k=0} \frac{(1-(\sigma\tau)^{m+1}(\rho\tau)^{k})(1-(\rho\tau)^{k+1}(\sigma\tau)^{m})}{(1-(\sigma\tau)^{m+1}(\rho\tau)^{k+1})^{2}} \quad \Gamma(\tau^{2};\sigma\tau,\rho\tau)^{-1}$$

$$= \Gamma(\tau^{2};\sigma\tau,\rho\tau)^{-1} \prod_{k,m=0}^{\infty} (1-(\sigma\tau)^{m+1})(1-(\rho\tau)^{k+1})$$

$$\equiv \frac{(\sigma\tau;\sigma\tau)(\rho\tau;\rho\tau)}{\Gamma(\tau^{2};\sigma\tau,\rho\tau)}$$

Pochhammer symbol

$$(x;p) := \prod_{\ell=0}^{\infty}(1-xp^{\ell})$$

2) For $\tilde{Q}_{2\dot{-}}$: we have that the states satisfy:

$$\tilde{\delta}_{2\dot{-}} = \Delta - 2j_2 + 2R + r = 0$$ ← This is the same as the one for $\tilde{Q}_{\dot{-}}$ with $R \to -R$

The $\mathcal{N}=4$ SYM theory is $\mathcal{N}=2$ with one vector and one adjoint hyper

The δ's that commute with $\tilde{Q}_{2\dot{-}}$ and $\tilde{S}^{2\dot{-}}$ are $\tilde{\delta}_{1\dot{+}}, \delta_{2+}, \delta_{2-}$ so we can define the index through:

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}(-1)^{F} e^{-\beta\tilde{\delta}_{2\dot{-}}}\, \tilde{\sigma}^{\frac{1}{2}\tilde{\delta}_{1\dot{+}}}\, \tilde{\rho}^{\frac{1}{2}\delta_{2+}}\, \tilde{\tau}^{\frac{1}{2}\delta_{2-}}$$

Vector contributions will be: (from $\phi, \bar{\phi}, \lambda_{I\alpha}, \bar{\lambda}_{I\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$)

	Δ	j_1	j_2	R	r	$\mathcal{I}$
ϕ	1	0	0	0	-1	$\tilde{\rho}\tilde{\tau}$
$\bar{F}_{\dot{+}\dot{+}}$	2	0	1	0	0	$\tilde{\rho}\,\tilde{\sigma}^{2}\tilde{\tau}$
$\lambda_{2\pm}$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\tilde{\rho}\,\tilde{\sigma}\,,\ -\tilde{\sigma}\tilde{\tau}$
$\bar{\lambda}_{2\dot{+}}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\tilde{\sigma}^{2}$
$\partial_{+\dot{+}}\lambda_{2-} + \partial_{-\dot{+}}\lambda_{2+}$	$\frac{5}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\rho}\,\tilde{\sigma}^{2}\tilde{\tau}$

For the $\frac{1}{2}$-hyper : (from $q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot\alpha}$)

$$\left\{ \begin{array}{ccccccc} & \Delta & j_1 & j_2 & R & r & \mathcal{I} \\ \bar{q} & 1 & 0 & 0 & -\frac{1}{2} & 0 & \tilde{\sigma} \\ \bar{\psi}_{\dot +} & \frac{3}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\tilde{\rho}\tilde{\sigma}\tilde{\tau} \end{array} \right.$$

and for the derivatives : (from $\partial_{\alpha\dot\alpha}$)

$$\left\{ \begin{array}{ccccccc} & \Delta & j_1 & j_2 & R & r & \mathcal{I} \\ \partial_{\pm\dot +} & 1 & \pm\frac{1}{2} & \frac{1}{2} & 0 & 0 & \tilde{\rho}\tilde{\sigma},\ \tilde{\sigma}\tilde{\tau} \end{array} \right.$$

Thus for the vector single-letter index we have :

$$i_{vec} = \frac{\tilde{\rho}\tilde{\tau} + 2\tilde{\rho}\tilde{\sigma}^2\tilde{\tau} - \tilde{\sigma}^2 - \tilde{\rho}\tilde{\sigma} - \tilde{\sigma}\tilde{\tau}}{(1-\tilde{\rho}\tilde{\sigma})(1-\tilde{\sigma}\tilde{\tau})} = -\frac{\tilde{\rho}\tilde{\sigma}}{(1-\tilde{\rho}\tilde{\sigma})} - \frac{\tilde{\sigma}\tilde{\tau}}{(1-\tilde{\sigma}\tilde{\tau})} + \frac{\tilde{\rho}\tilde{\tau} - \tilde{\sigma}^2}{(1-\tilde{\rho}\tilde{\sigma})(1-\tilde{\sigma}\tilde{\tau})}$$

while for the $\frac{1}{2}$ hyper :

$$i_{\frac{1}{2}H} = \frac{\tilde{\sigma} - \tilde{\rho}\tilde{\sigma}\tilde{\tau}}{(1-\tilde{\sigma}\tilde{\tau})(1-\tilde{\rho}\tilde{\sigma})} = \frac{\tilde{\sigma}(1-\tilde{\rho}\tilde{\tau})}{(1-\tilde{\sigma}\tilde{\tau})(1-\tilde{\rho}\tilde{\sigma})}$$

These are identical to the contributions we had from the $\tilde{Q}_{1-}$ definition of the index upon performing the redefinitions :

$$\tilde{\sigma} \to \tau, \quad \tilde{\rho} \to \sigma, \quad \tilde{\tau} \to \rho$$

Now the "$\mathcal{N}=2$" index over $\mathcal{N}=4$ SYM is just :

$$\mathcal{I}^{\mathcal{N}=4} = \int [dU]\, P.E.\left[(i_v + i_H)\, \chi_{adj}(U)\right]$$

and only depends on the $i_{vec}, i_{\frac{1}{2}H}$ so it will be the same for both $\tilde{Q}_{1-}, \tilde{Q}_{2-}$.