$\mathcal{CPT}$ symmetry in quantum field theory

Andreas Fring

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Outline

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4. Parity, charge conjugation, time reversal, strong reflection, Hermitian conjugation
5. $\mathcal{CPT}$ versus strong reflection + Hermitian conjugation
6. Invariant interaction terms
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Motivation

- The $\mathcal{CPT}$ theorem is based on the cornerstones of physics: standard principles of quantum mechanics, Lorentz invariance, energy positivity, causality
  $\Rightarrow$ test of these foundations
- model building:
  What kind of terms are permissible in a consistent Hamiltonian?
  - quantum mechanics $\mathcal{PT}$-symmetry
  - quantum field theory $\mathcal{CPT}$-symmetry
- $\Rightarrow$ models beyond the Standard Model with $\mathcal{CPT}$-violation?
- matter/anti-matter balance is broken in the universe

Google scholar articles with "$\mathcal{CPT}$" in the title 13200
Google scholar articles with "$\mathcal{CP}$ violation in the title 10500
Google scholar articles with "time reversal" in the title 283000

$\mathcal{PT}$-symmetry in quantum mechanics

Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:

$$\mathcal{PT}(\lambda \Phi + \mu \Psi) = \lambda^* \mathcal{PT} \Phi + \mu^* \mathcal{PT} \Psi \qquad \lambda, \mu \in \mathbb{C}$$

- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \quad \Rightarrow \varepsilon = \varepsilon^* \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$

- *Proof*: $\varepsilon\Phi = \mathcal{H}\Phi = \mathcal{HPT}\Phi = \mathcal{PTH}\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi$

Spontaneously broken $\mathcal{PT}$-symmetry gives conjugate eigenvalues

- Spontaneously broken $\mathcal{PT}$-symmetry:

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi \neq \Phi$$

- Instead

$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi_1 = \Phi_2$$

$$\mathcal{H}\Phi_1 = \varepsilon_1\Phi_1 \qquad \mathcal{H}\Phi_2 = \varepsilon_2\Phi_2$$

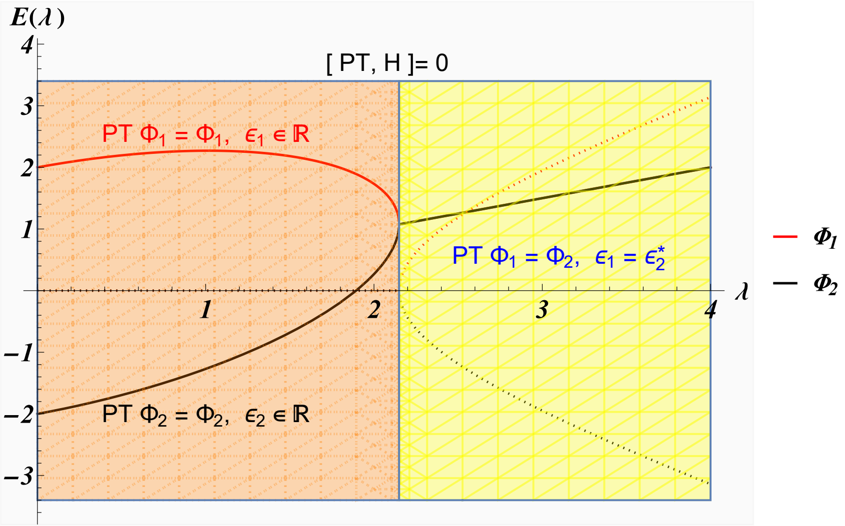
$$\Rightarrow \mathcal{PTH}\Phi_1 = \mathcal{PT}\varepsilon_1\Phi_1 \Rightarrow \mathcal{HPT}\Phi_1 = \varepsilon_1^*\mathcal{PT}\Phi_1 \Rightarrow \mathcal{H}\Phi_2 = \varepsilon_1^*\Phi_2$$

The eigenvalues of Φ_1 and Φ_2 form a complex conjugate pair.

- The point in parameter space where the $\mathcal{PT}$-symmetry spontaneously breaks is referred to as exceptional point.

$\mathcal{PT}$-symmetry is only an example of an antilinear operator.

$\mathcal{PT}$-symmetry versus spontaneously broken $\mathcal{PT}$-symmetry



real parts are solid lines, imaginary parts are dotted lines

H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \eta^\dagger \eta = \eta^\dagger \eta H$$

$$\Phi = \eta^{-1}\phi \qquad \eta^\dagger = \eta$$

$\Rightarrow$ H is Hermitian with respect to the new metric

Proof:

$$\langle \Psi | H\Phi \rangle_\eta = \langle \Psi | \eta^2 H\Phi \rangle = \langle \eta^{-1}\psi | \eta^2 H \eta^{-1}\phi \rangle = \langle \psi | \eta H \eta^{-1}\phi \rangle =$$

$$\langle \psi | h\phi \rangle = \langle h\psi | \phi \rangle = \langle \eta H \eta^{-1}\psi | \phi \rangle = \langle H\Psi | \eta\phi \rangle = \left\langle H\Psi | \eta^2 \Phi \right\rangle$$

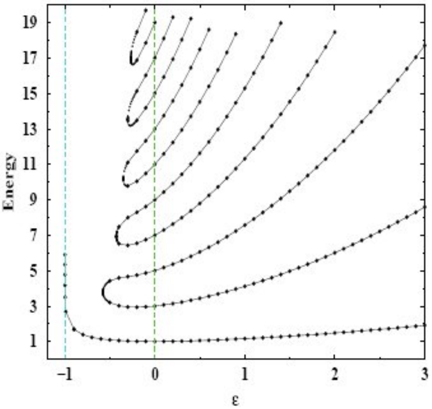
$$= \langle H\Psi | \Phi \rangle_\eta$$

Using the same reasoning as in the Hermitian case:

$\Rightarrow$ Eigenvalues of H are real, eigenstates are orthogonal

The seminal classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^\varepsilon \qquad \text{for } \varepsilon \in \mathbb{R}$$



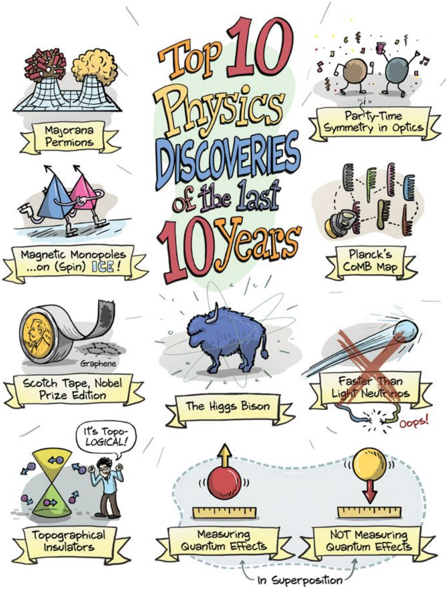
- real eigenvalues for $\varepsilon \geq 0$
- exceptional points for $\varepsilon < 0$
- Try for yourself:

$$H = -\frac{1}{2}\left[\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right]$$

with $\omega, \lambda, \kappa \in \mathbb{R}$

C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243

Nature Physics volume 11, page 799 (2015)



Helmholtz equation
in paraxial approximation:

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2k}\frac{\partial^2\psi}{\partial x^2} + kv(x)\psi = 0$$

$\psi \equiv$ envelope function of E

$v(x) = n/n_0 - 1$

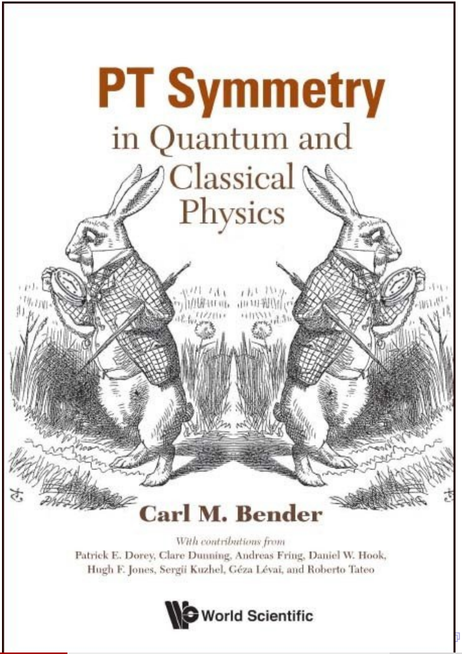
$n \equiv$ reflection index

$n_0 \equiv$ reflection index

$k = n\omega/c$

$\omega \equiv$ frequency

with $z \to t$
this becomes formally
the Schrödinger equation


PT Symmetry
in Quantum and
Classical
Physics
Carl M. Bender
With contributions from
Patrick E. Dorey, Clare Dunning, Andreas Fring, Daniel W. Hook,
Hugh F. Jones, Sergii Kuzhel, Géza Lévai, and Roberto Tateo
World Scientific

$\mathcal{CPT}$ theorem

> Every relativistic quantum field theory is invariant under a simultaneous change of particles into antiparticles (Charge conjugation $\mathcal{C}$), reflection about some arbitrary point in space (Parity $\mathcal{P}$) and the reverse of the direction of time (Time reversal $\mathcal{T}$).

Here we proof this for spin 0, 1/2 and 1 fields with covariant interaction terms. The strategy of the proof is to first establish the invariance for the field equations in the interaction representation and demanding that the free field commutation relations are preserved. Subsequently we investigate the invariance of covariant interaction terms.

J. Schwinger, Phys. Rev. 82 (1951) 914
G. Lüders, Matematisk-Fysike Meddelelser 28.5 (1954).
G. Lüders, Annals of Physics 2.1 (1957): 1-15.
W. Pauli, Il Nuovo Cimento (1955-1965) 6.1 (1957): 204-215.
R. Jost, Helv. Phys. Acta 30.409 (1957): 153.
F.J. Dyson, Phys. Rev. 110 (1958): 579.

Field equations, commutation relations (interaction representation)

Spin 0 field $\phi(x)$:

$$\left(\partial_\mu \partial^\mu - m^2\right)\phi(x) = 0, \quad \left(\partial_\mu \partial^\mu - m^2\right)\phi^*(x) = 0,$$

$$[\phi(x), \phi(y)] = [\phi^*(x), \phi^*(y)] = 0, \quad [\phi(x), \phi^*(y)] = -i\Delta(x-y)$$

Spin 1 field $\varphi_\mu(x)$:

$$\begin{aligned}
\left(\partial_\mu \partial^\mu - m^2\right)\varphi_\mu(x) &= 0, \quad \left(\partial_\mu \partial^\mu - m^2\right)\varphi_\mu^*(x) = 0, \\
\partial^\mu \varphi_\mu(x) &= \partial^\mu \varphi_\mu^*(x) = 0 \\
\left[\varphi_\mu(x), \varphi_\nu(y)\right] &= \left[\varphi_\mu^*(x), \varphi_\nu^*(y)\right] = 0, \\
\left[\varphi_\mu^*(x), \varphi_\nu(y)\right] &= -i\left(g_{\mu\nu} - m^{-2}\right)\Delta(x-y)
\end{aligned}$$

We use a Lorentzian metric: $-g_{00} = g_{11} = g_{22} = g_{33} = 1$.

Field equations, commutation relations (interaction representation)

Spin 1/2 field $\psi(x)$:

$$\begin{aligned}(\gamma_\mu \partial^\mu + m)\,\psi(x) &= 0, \quad \bar{\psi}(x)\,(\gamma_\mu \partial^\mu - m) = 0, \\ \{\psi_\alpha(x), \psi_\beta(y)\} &= \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = 0, \\ \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} &= i\,(\gamma_\mu \partial^\mu - m)_{\alpha\beta}\,\Delta(x-y)\end{aligned}$$

where

$$\begin{aligned}\bar{\psi}(x) &:= -i\psi^\dagger(x)\gamma_0 \\ \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu}, \quad \gamma_0^\dagger = -\gamma_0, \;\; \gamma_k^\dagger = \gamma_k \;\; (k = 1,2,3), \\ \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \{\gamma_5, \gamma_\mu\} = 0\end{aligned}$$

further properties

$$\Delta(\vec{x}, t) = \Delta(-\vec{x}, t) = -\Delta(\vec{x}, -t) = -\Delta(-\vec{x}, -t)$$

e.g. spin 0: $$\Delta(x-y) = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{k_0} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sin\left[k_0(x_0 - y_0)\right]$$

Parity $\mathcal{P}$

Reflect entire physics about an arbitrary point in space, e.g. the origin

$$
\begin{aligned}
\mathcal{P}\phi(\vec{x},t)\mathcal{P}^{-1} &= \eta_P\phi(-\vec{x},t), & \mathcal{P}\phi^*(\vec{x},t)\mathcal{P}^{-1} &= \eta_P^*\phi^*(-\vec{x},t),\\
\mathcal{P}\varphi_k(\vec{x},t)\mathcal{P}^{-1} &= -\hat{\eta}_P\varphi_k(-\vec{x},t), & \mathcal{P}\varphi_k^*(\vec{x},t)\mathcal{P}^{-1} &= -\hat{\eta}_P^*\varphi_k^*(-\vec{x},t),\\
\mathcal{P}\varphi_0(\vec{x},t)\mathcal{P}^{-1} &= \hat{\eta}_P\varphi_0(-\vec{x},t), & \mathcal{P}\varphi_0^*(\vec{x},t)\mathcal{P}^{-1} &= \hat{\eta}_P^*\varphi_0^*(-\vec{x},t),\\
\mathcal{P}\psi(\vec{x},t)\mathcal{P}^{-1} &= \check{\eta}_P\gamma_0\psi(-\vec{x},t), & \mathcal{P}\bar{\psi}(\vec{x},t)\mathcal{P}^{-1} &= -\check{\eta}_P^*\bar{\psi}(-\vec{x},t)\gamma_0,
\end{aligned}
$$

with

$$\mathcal{P}\mathcal{P}^\dagger = \mathbb{I}, \qquad |\eta_P| = |\hat{\eta}_P| = |\check{\eta}_P| = 1$$

- adointness is preserved
- for Hermitian fields we have $\eta_P = \pm 1$

$\underline{\underline{0}}$ $(\partial_\mu \partial^\mu - m^2)\,\phi(\vec{x}, t) = 0 \xrightarrow{\vec{x} \to -\vec{x}} (\partial_\mu \partial^\mu - m^2)\,\underbrace{\eta_p\,\phi(-\vec{x}, t)}_{P\,\phi(\vec{x}, t)\,P^{-1}} = 0$

$[\phi(\vec{x}, t), \phi^*(\vec{y}, t')] = -i\,\Delta(\vec{x} - \vec{y}, t - t')$

$P:$ $[\phi(-\vec{x}, t), \phi^*(-\vec{y}, t')]\,\underbrace{\eta_p \eta_p^*}_{1} = \underbrace{-i\,(-\vec{x} + \vec{y}, t - t')}_{\Delta(-\vec{r}, t) = \Delta(\vec{r}, t)}$

$\underline{1}$ $\mu = 0$, $\mu = 1, 2, 3$

$\underline{\tfrac{1}{2}}$: $(\gamma_\mu \partial^\mu + m)\,\psi(\vec{x}, t) = 0$

$(-\gamma_0 \partial_0 + \gamma_k \partial_k + m)\,\psi(\vec{x}, t) = 0$ $k = 1, 2, 3$

$\vec{x} \to -\vec{x}\,\big|\ \gamma_0(-\gamma_0 \partial_0 - \gamma_k \partial_k + m)\,\psi(-\vec{x}, t) = 0$

$\gamma_0\,\big|\ (-\gamma_0 \partial_0 + \gamma_k \partial_k + m)\,\gamma_0\,\psi(-\vec{x}, t) = 0$ $\underline{\{\gamma_\mu, \gamma_0\} = 2g_{\mu 0}}$

$(\gamma_\mu \partial^\mu + m)\,\underbrace{\eta_p\,\gamma_0\,\psi(-\vec{x}, t)}_{P\,\psi(\vec{x}, t)\,P^{-1}} = 0$

Charge conjugation $\mathcal{C}$

Change particles to antiparticles, operators transform to their adjoints

$$\mathcal{C}\phi(\vec{x},t)\mathcal{C}^{-1} = \eta_C \phi^*(\vec{x},t), \qquad \mathcal{C}\phi^*(\vec{x},t)\mathcal{C}^{-1} = \eta_C^* \phi(\vec{x},t),$$

$$\mathcal{C}\varphi_\mu(\vec{x},t)\mathcal{C}^{-1} = \hat{\eta}_C \varphi_\mu^*(\vec{x},t), \qquad \mathcal{C}\varphi_\mu^*(\vec{x},t)\mathcal{C}^{-1} = \hat{\eta}_C^* \varphi_\mu^*(\vec{x},t),$$

$$\mathcal{C}\psi(\vec{x},t)\mathcal{C}^{-1} = \check{\eta}_C C\bar{\psi}^\intercal(\vec{x},t), \qquad \mathcal{C}\bar{\psi}(\vec{x},t)\mathcal{C}^{-1} = -\check{\eta}_C^* \psi^\intercal(\vec{x},t) C^\dagger,$$

with

$$CC^\dagger = \mathbb{I}, \qquad C\gamma_\mu^\intercal = -\gamma_\mu C, \qquad C^\intercal = -C$$

$$|\eta_C| = |\hat{\eta}_C| = |\check{\eta}_C| = 1$$

Time reversal $\mathcal{T}$

Reverse time of each physical state, velocities are reversed (Wigner)

$$\begin{aligned}
\mathcal{T}\phi(\vec{x},t)\mathcal{T}^{-1} &= \eta_T\phi(\vec{x},-t), & \mathcal{T}\phi^*(\vec{x},t)\mathcal{T}^{-1} &= \eta_T^*\phi^*(\vec{x},-t),\\
\mathcal{T}\varphi_k(\vec{x},t)\mathcal{T}^{-1} &= \hat{\eta}_T\varphi_k(\vec{x},-t), & \mathcal{T}\varphi_k^*(\vec{x},t)\mathcal{T}^{-1} &= \hat{\eta}_T^*\varphi_k^*(\vec{x},-t),\\
\mathcal{T}\varphi_0(\vec{x},t)\mathcal{T}^{-1} &= -\hat{\eta}_T\varphi_0(\vec{x}-,t), & \mathcal{T}\varphi_0^*(\vec{x},t)\mathcal{T}^{-1} &= -\hat{\eta}_T^*\varphi_0^*(\vec{x},-t),\\
\mathcal{T}\psi(\vec{x},t)\mathcal{T}^{-1} &= \check{\eta}_T T\psi(\vec{x},-t), & \mathcal{T}\bar{\psi}(\vec{x},t)\mathcal{T}^{-1} &= \check{\eta}_T^*\bar{\psi}(\vec{x},-t)T,
\end{aligned}$$

$$TT^\dagger = \mathbb{I}, \qquad \gamma_\mu^\intercal T = T\gamma_\mu, \qquad T^\intercal = T \qquad \mathcal{T}\lambda\mathcal{T}^{-1}=\lambda^* \quad \lambda\in\mathbb{C}$$
$$|\eta_T| = |\hat{\eta}_T| = |\check{\eta}_T| = 1$$

Combined $\mathcal{CT}$:
$C\gamma_\mu^\intercal C^{-1} = -\gamma_\mu$, $\gamma_\mu^\intercal = T\gamma_\mu T^{-1} \Rightarrow CT\gamma_\mu,(CT)^{-1} = -\gamma_\mu$
Therefore we may set $\mathcal{CT} := \gamma_5$

Simplified notation

Instead of

$$\mathcal{CPT}\mathcal{O}(x)\mathcal{CPT}^{-1} = \mathcal{CP}\mathcal{O}'(x)\mathcal{CP}^{-1} = \mathcal{C}\mathcal{O}''(x)\mathcal{C}^{-1} = \mathcal{O}'''(x)$$

we write

$$\mathcal{O}(x) \xrightarrow{\mathcal{T}} \mathcal{O}'(x) \xrightarrow{\mathcal{P}} \mathcal{O}''(x) \xrightarrow{\mathcal{C}} \mathcal{O}'''(x)$$

and we replace

$$\mathcal{CPT}\mathcal{O}(x)\mathcal{CPT}^{-1} = \mathcal{O}'''(x)$$

by

$$\mathcal{O}(x) \xrightarrow{\mathcal{CPT}} \mathcal{O}'''(x)$$

Combined $\mathcal{CPT}$ transformation

$$\phi(\vec{x},t) \xrightarrow{\mathcal{T}} \eta_T \phi(\vec{x},-t) \xrightarrow{\mathcal{P}} \eta_T \eta_P \phi(-\vec{x},-t) \xrightarrow{\mathcal{C}} \eta_T \eta_P \eta_C \phi^*(-\vec{x},-t)$$

$$\varphi_k(\vec{x},t) \xrightarrow{\mathcal{T}} \hat{\eta}_T \varphi_k(\vec{x},-t) \xrightarrow{\mathcal{P}} -\hat{\eta}_T \hat{\eta}_P \varphi_k(-\vec{x},-t) \xrightarrow{\mathcal{C}} -\hat{\eta}_T \hat{\eta}_P \hat{\eta}_C \hat{\varphi}_k^*(-\vec{x},-t)$$

$$\varphi_0(\vec{x},t) \xrightarrow{\mathcal{T}} -\hat{\eta}_T \varphi_0(\vec{x},-t) \xrightarrow{\mathcal{P}} -\hat{\eta}_T \hat{\eta}_P \varphi_0(-\vec{x},-t) \xrightarrow{\mathcal{C}} -\hat{\eta}_T \hat{\eta}_P \hat{\eta}_C \hat{\varphi}_0^*(-\vec{x},-t)$$

$$\psi(\vec{x},t) \xrightarrow{\mathcal{T}} \check{\eta}_T T \psi(\vec{x},-t) \xrightarrow{\mathcal{P}} \hat{\eta}_T \hat{\eta}_P T \gamma_0 \psi(-\vec{x},-t)$$

$$\xrightarrow{\mathcal{C}} \check{\eta}_T \check{\eta}_P \check{\eta}_C C T \gamma_0 \bar{\psi}^\intercal(-\vec{x},-t)$$

Therefore (with the same logic for the conjugate fields)

$$\phi(x) \xrightarrow{\mathcal{CPT}} \eta_T \eta_P \eta_C \phi^*(-x) \qquad \phi^*(x) \xrightarrow{\mathcal{CPT}} \eta_T \eta_P \eta_C \phi(-x)$$

$$\varphi_\mu(x) \xrightarrow{\mathcal{CPT}} -\hat{\eta}_T \hat{\eta}_P \hat{\eta}_C \varphi_\mu^*(-x) \qquad \varphi_\mu^*(x) \xrightarrow{\mathcal{CPT}} -\hat{\eta}_T \hat{\eta}_P \hat{\eta}_C \varphi_\mu(-x)$$

$$\psi(x) \xrightarrow{\mathcal{CPT}} \check{\eta}_T \check{\eta}_P \check{\eta}_C \gamma_5 \gamma_0 \bar{\psi}^\intercal(-x) \qquad \bar{\psi}(x) \xrightarrow{\mathcal{CPT}} \check{\eta}_T \check{\eta}_P \check{\eta}_C \psi^\intercal(-x) \gamma_5 \gamma_0$$

$$\lambda \xrightarrow{\mathcal{CPT}} \lambda^*$$

Strong reflection

Simultaneous reflection of time, space and the reversal of the order of the field operators (Pauli)

$$\phi(\vec{x},t) \to \phi(-\vec{x},-t), \qquad \phi^*(\vec{x},t) \to \phi^*(-\vec{x},-t),$$
$$\varphi_\mu(\vec{x},t) \to -\varphi_\mu(-\vec{x},-t), \qquad \varphi_\mu^*(\vec{x},t) \to -\varphi_\mu^*(-\vec{x},-t),$$
$$\psi(\vec{x},t) \to i\gamma_5\psi(-\vec{x},-t), \qquad \bar{\psi}^*(\vec{x},t) \to i\bar{\psi}^*(-\vec{x},-t)\gamma_5,$$

- Interaction terms must be symmetrized with respect to Bose field and antisymmetrized with respect to Fermi field, i.e. Wick ordering :: is applied
- strong reflection is only defined for the operator algebra
- strong reflection can not be applied on the Hilbert space

Hermitian conjugation

Simultaneously conjugate field operators, reverse their order and conjugate c numbers $\lambda \to \lambda^*$, $\lambda \in \mathbb{C}$

$$\begin{aligned}
&\phi(\vec{x}, t) \leftrightarrow \phi^*(\vec{x}, t), \\
&\varphi_\mu(\vec{x}, t) \leftrightarrow \varphi^*_\mu(\vec{x}, t), \\
&\psi(\vec{x}, t) \leftrightarrow \psi^*(\vec{x}, t) = -i\bar{\psi}^\intercal(\vec{x}, t)\gamma_0
\end{aligned}$$

Strong reflection + Hermitian conjugation

$$\phi(\vec{x},t) \to \phi^*(-\vec{x},-t), \qquad \phi^*(\vec{x},t) \to \phi(-\vec{x},-t),$$
$$\varphi_\mu(\vec{x},t) \to -\varphi_\mu^*(-\vec{x},-t), \qquad \varphi_\mu^*(\vec{x},t) \to -\varphi_\mu(-\vec{x},-t),$$
$$\psi(\vec{x},t) \to \gamma_5\gamma_0\bar{\psi}^\intercal(-\vec{x},-t), \qquad \bar{\psi}(\vec{x},t) \to \psi^\intercal(-\vec{x},-t)\gamma_5\gamma_0,$$
$$\lambda \xrightarrow{\mathcal{CPT}} \lambda^*$$

- The order of all operators remains the same.
- The combined transformation is antilinear since $\lambda \to \lambda^*$.
- The combined transformation can be applied to the operator algebra and the Hilbert space.

$\mathcal{CPT}$ versus Strong reflection + Hermitian conjugation

We observe that the combined $\mathcal{CPT}$ transformation coincides exactly with the combined transformation of strong reflection + Hermitian conjugation when we take

$$\eta_T \eta_P \eta_C = \hat{\eta}_T \hat{\eta}_P \hat{\eta}_C = \check{\eta}_T \check{\eta}_P \check{\eta}_C = 1$$

Interaction terms

Hermitian interaction terms must therefore transform as

$$\mathcal{H}(\vec{x},t) \xrightarrow{\mathcal{CPT}} \mathcal{H}(-\vec{x},-t)$$

Consider Lorentz invariant terms, e.g. for spin 1/2-operators

$$\bar{\psi}\psi, \quad i\bar{\psi}\gamma_\mu\psi, \quad \bar{\psi}\gamma_\mu\gamma_\nu\psi, \quad i\bar{\psi}\gamma_\mu\gamma_5\psi, \quad i\bar{\psi}\gamma_5\psi,$$

Examples:

Tensors $\Phi_{\mu_1\ldots\mu_n}(x)$ composed of $\partial_\mu\phi(x)$ and $\varphi_\mu(x)$ transform as

$$\Phi_{\mu_1\ldots\mu_n}(x) \xrightarrow{\mathcal{CPT}} (-1)^n \Phi^\dagger_{\mu_1\ldots\mu_n}(-x)$$

Bilinears of Dirac fields transform as

$$\begin{aligned} \bar{\psi}(x)\psi(x) &= \left[\bar{\psi}(x)\psi(x)\right]^\dagger \xrightarrow{\mathcal{CPT}} \psi^\intercal(-x)\gamma_5\gamma_0\gamma_5\gamma_0\bar{\psi}^\intercal(-x) \\ &= \bar{\psi}(-x)\psi(-x) \end{aligned}$$

$$
\begin{aligned}
i\bar{\psi}(x)\gamma_0\psi(x) &= \left[i\bar{\psi}(x)\gamma_0\psi(x)\right]^\dagger \overset{\mathcal{CPT}}{\longrightarrow} -i\psi^\intercal(-x)\gamma_5\gamma_0\gamma_0^*\gamma_5\gamma_0\bar{\psi}^\intercal(-x) \\
&= -i\bar{\psi}(-x)\gamma_0\psi(-x) \\
i\bar{\psi}(x)\gamma_5\psi(x) &= \left[i\bar{\psi}(x)\gamma_0\psi(x)\right]^\dagger \overset{\mathcal{CPT}}{\longrightarrow} -i\psi^\intercal(-x)\gamma_5\gamma_0\gamma_5^*\gamma_5\gamma_0\bar{\psi}^\intercal(-x) \\
&= i\bar{\psi}(-x)\gamma_5\psi(-x)
\end{aligned}
$$

What about separate transformations?

- QED and QCD are invariant under the separate $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations.
- Chiral interactions violate separate $\mathcal{C}$ and $\mathcal{P}$ transformations, but are invariant under the combined $\mathcal{CP}$ transformation

Consider the chiral interaction term:

$$\mathcal{L}_{ch} = \bar{\psi}\gamma_\mu(1-\gamma_5)\psi A^\mu$$

Charge conjugation (ignore the arguments)

$$
\begin{aligned}
\bar{\psi}\gamma_\mu\psi A^\mu \xrightarrow{\mathcal{C}} & \quad -\eta_C\eta_C^*\psi^\intercal C^\dagger\gamma_\mu C\bar{\psi}^\intercal A^\mu \\
& = -\psi^\intercal(-\gamma_\mu^\intercal)\bar{\psi}^\intercal A^\mu \\
& = \bar{\psi}\gamma_\mu\psi A^\mu \\
\bar{\psi}\gamma_\mu\gamma_5\psi A^\mu \xrightarrow{\mathcal{C}} & \quad -\eta_C\eta_C^*\psi^\intercal C^\dagger\gamma_\mu\gamma_5 C\bar{\psi}^\intercal A^\mu \\
& = -\psi^\intercal C^\dagger\gamma_\mu C C^\dagger\gamma_5 C\bar{\psi}^\intercal A^\mu \\
& = -\psi^\intercal(-\gamma_\mu^\intercal\gamma_5^\intercal)\bar{\psi}^\intercal A^\mu \\
& = \bar{\psi}\gamma_5\gamma_\mu\psi A^\mu \\
& = -\bar{\psi}\gamma_\mu\gamma_5\psi A^\mu
\end{aligned}
$$

therefore

$$
\mathcal{L}_{ch} \overset{\mathcal{C}}{\not\to} \mathcal{L}_{ch}
$$

Parity transformation

$$\bar{\psi}\gamma_{\mu}\psi A^{\mu} \xrightarrow{\mathcal{P}} -\eta_P\eta_P^*\bar{\psi}\gamma_0\gamma_{\mu}\gamma_0\psi(-A^{\mu})$$
$$= -\bar{\psi}\gamma_0\gamma_0\gamma_{\mu}\psi A^{\mu}$$
$$= \bar{\psi}\gamma_{\mu}\psi A^{\mu}$$
$$\bar{\psi}\gamma_{\mu}\gamma_5\psi A^{\mu} \xrightarrow{\mathcal{P}} -\eta_P\eta_P^*\bar{\psi}\gamma_0\gamma_{\mu}\gamma_5\gamma_0\psi(-A^{\mu})$$
$$= \bar{\psi}\gamma_0\gamma_0\gamma_{\mu}\gamma_5\psi A^{\mu}$$
$$= -\bar{\psi}\gamma_{\mu}\gamma_5\psi A^{\mu}$$

therefore

$$\mathcal{L}_{ch} \overset{\mathcal{P}}{\not\rightarrow} \mathcal{L}_{ch}$$

But

$$\mathcal{L}_{ch} \xrightarrow{\mathcal{CP}} \mathcal{L}_{ch}$$

$\mathcal{CPT}$ invariance, $\mathcal{CP}$ violation

$\mathcal{CPT}$ invariance

Lee and Yang[a] :

- Particles and antiparticles have equal lifetimes and masses.
- equal lifetimes $\Rightarrow$ either $\mathcal{CP}$ or $\mathcal{CPT}$ are valid
- different lifetimes $\Rightarrow$ both $\mathcal{CP}$ and $\mathcal{CPT}$ are violated

[a] Lee, T.-D., Yang C.-N. "Question of parity conservation in weak interactions." Physical Review 104.1 (1956): 254.

$\mathcal{CP}$ violation

Neutral Kaons: K^0, $\bar{K}^0$ mesons have spin 0 and opposite strangeness

$$K^0 \quad : \quad \phi_{K^0} := \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\bar{K}^0 \quad : \quad \phi_{\bar{K}^0} := \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

But these states are not $\mathcal{CP}$ eigenstates:

$$\mathcal{CP}\phi_{K^0} = \phi_{\bar{K}^0} \qquad \mathcal{CP}\bar{\phi}_{K^0} = \phi_{K^0}$$

Therefore define:

$$\phi_{K_1^0} := \frac{1}{\sqrt{2}}\left(\phi_{K^0} + \phi_{\bar{K}^0}\right) \qquad \phi_{K_2^0} := \frac{i}{\sqrt{2}}\left(\phi_{K^0} - \phi_{\bar{K}^0}\right)$$

such that

$$\mathcal{CP}\phi_{K_1^0} = \phi_{K_1^0} \qquad \mathcal{CP}\phi_{K_2^0} = -\phi_{K_2^0}$$

K mesons decay into π mesons for which we have

$$\mathcal{CP}\psi\left(\pi^+\pi^-\right) = \psi\left(\pi^+\pi^-\right) \qquad \mathcal{CP}\psi\left(\pi^+\pi^-\pi^0\right) = -\psi\left(\pi^+\pi^-\pi^0\right)$$

Therefore the following processes respect $\mathcal{CP}$ invariance

$$K_1^0 \rightarrow \pi^+\pi^-$$
$$K_2^0 \rightarrow \pi^+\pi^-\pi^0$$

However, in an experiment reported in [Christenson, J. H., et al. "Evidence for the π Decay of the K_2^0 Meson." Physical Review Letters 13.4 (1964): 138] the authors found the following two pion decay

$$K_2^0 \rightarrow \pi^+\pi^-$$

which violates the $\mathcal{CP}$-invariance.
Assuming that $\mathcal{CPT}$ holds this implies that this process is also not invariant with respect to time reversal $\mathcal{T}$.

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