LonTI Lecture on the Superconformal Index

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Consider the following assignment of quantum numbers for the Poincaré supercharges of the 4D $\mathcal{N} = 2$ SCA:

$\mathcal{Q}$	$SU(2)_1$	$SU(2)_2$	$SU(2)_R$	$U(1)_r$	$\delta = 2\{\mathcal{Q}, \mathcal{Q}^\dagger\}$	Commuting δs
$\mathcal{Q}_{1-}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\delta_{1-} = \Delta - 2j_1 - 2R - r$	$\delta_{2+}, \quad \tilde{\delta}_{1\dot{+}}, \quad \tilde{\delta}_{1\dot{-}}$
$\mathcal{Q}_{1+}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\delta_{1+} = \Delta + 2j_1 - 2R - r$	$\delta_{2-}, \quad \tilde{\delta}_{1\dot{+}}, \quad \tilde{\delta}_{1\dot{-}}$
$\mathcal{Q}_{2-}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\delta_{2-} = \Delta - 2j_1 + 2R - r$	$\delta_{1+}, \quad \tilde{\delta}_{2\dot{+}}, \quad \tilde{\delta}_{2\dot{-}}$
$\mathcal{Q}_{2+}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\delta_{2+} = \Delta + 2j_1 + 2R - r$	$\delta_{1-}, \quad \tilde{\delta}_{2\dot{+}}, \quad \tilde{\delta}_{2\dot{-}}$
$\tilde{\mathcal{Q}}_{1\dot{-}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{1\dot{-}} = \Delta - 2j_2 - 2R + r$	$\tilde{\delta}_{2\dot{+}}, \quad \delta_{1+}, \quad \delta_{1-}$
$\tilde{\mathcal{Q}}_{1\dot{+}}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{1\dot{+}} = \Delta + 2j_2 - 2R + r$	$\tilde{\delta}_{2\dot{-}}, \quad \delta_{1+}, \quad \delta_{1-}$
$\tilde{\mathcal{Q}}_{2\dot{-}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{2\dot{-}} = \Delta - 2j_2 + 2R + r$	$\tilde{\delta}_{1\dot{+}}, \quad \delta_{2+}, \quad \delta_{2-}$
$\tilde{\mathcal{Q}}_{2\dot{+}}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\tilde{\delta}_{2\dot{+}} = \Delta + 2j_2 + 2R + r$	$\tilde{\delta}_{1\dot{-}}, \quad \delta_{2+}, \quad \delta_{2-}$

1. Evaluate the single-letter index over the free vector multiplet and the free (half)hypermultiplet. The letters contributing to the "$\mathcal{N} = 2$ index" with respect to $\tilde{\mathcal{Q}}_{1\dot{-}}$, as defined in the lecture, comprise:

Letters	Δ	j_1	j_2	R	r
ϕ	1	0	0	0	-1
$\lambda_{1\pm}$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
$\bar{\lambda}_{1\dot{+}}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{F}_{\dot{+}\dot{+}}$	2	0	1	0	0
$\partial_{-\dot{+}}\lambda_{1+} + \partial_{+\dot{+}}\lambda_{1-} = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
q	1	0	0	$\frac{1}{2}$	0
$\bar{\psi}_{+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\partial_{\pm+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0

Then carry out the plethystic exponentials for each case.

2. Define the "$\mathcal{N}=2$ index" with respect to the $\tilde{\mathcal{Q}}_{2\dot{-}}$ supercharge. Argue that this gives the same answer as the one defined with respect to $\tilde{\mathcal{Q}}_{1\dot{-}}$ for 4D $\mathcal{N}=4$ SYM with $U(N)$ gauge group.

3. Write the Schur limit of the $\mathcal{N}=2$ index defined in the lecture as

$$\mathcal{I}_S = \mathrm{Tr}(-1)^F \rho^{2(\Delta-R)}$$

and evaluate it over the free hypermultiplet and vector multiplet.

4. The short multiplets of the 4D $\mathcal{N}=2$ SCA that contribute to the index as defined in the lecture are:

$$\begin{aligned}
\mathcal{I}_{\overline{\mathcal{C}}_{R,r(j_1,j_2)}} =& -(-1)^{2(j_1+j_2)}\tau^{2+2R+2j_2}\sigma^{j_2-r}\rho^{j_2-r}\frac{(1-\sigma\rho)(\tau-\sigma)(\tau-\rho)}{(1-\sigma\tau)(1-\rho\tau)}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\\
\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} =&(-1)^{2(j_1+j_2)}\frac{\tau^{3+2R+2j_2}\sigma^{j_1+\frac{1}{2}}\rho^{j_1+\frac{1}{2}}(1-\sigma\rho)}{(1-\sigma\tau)(1-\rho\tau)}\left(\chi_{2j_1+1}\left(\sqrt{\frac{\sigma}{\rho}}\right)-\frac{\sqrt{\sigma\rho}}{\tau}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\right)\\
\mathcal{I}_{\overline{\mathcal{E}}_{r(j_1,0)}} =&(-1)^{2j_1}\sigma^{-r-1}\rho^{-r-1}\frac{(\tau-\sigma)(\tau-\rho)}{(1-\sigma\tau)(1-\rho\tau)}\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right),\\
\mathcal{I}_{\overline{\mathcal{D}}_{0(j_1,0)}} =&\frac{(-1)^{2j_1}(\sigma\rho)^{j_1+1}}{(1-\sigma\tau)(1-\rho\tau)}\times\\
&\left(\left(1+\tau^2\right)\chi_{2j_1}\left(\sqrt{\frac{\sigma}{\rho}}\right)-\frac{\tau}{\sqrt{\sigma\rho}}\chi_{2j_1+1}\left(\sqrt{\frac{\sigma}{\rho}}\right)-\tau\sqrt{\sigma\rho}\chi_{2j_1-1}\left(\sqrt{\frac{\sigma}{\rho}}\right)\right)\\
\mathcal{I}_{\mathcal{D}_{0(0,j_2)}} =&\frac{(-1)^{2j_2+1}\tau^{2j_2+2}}{(1-\sigma\tau)(1-\rho\tau)}(1-\sigma\rho)\ .
\end{aligned}$$

The subscripts denote various short superconformal multiplets of the $\mathcal{N}=2$ SCA, satisfying particular shortening conditions and labelled by the quantum numbers of the superconformal primary. The Schur polynomial $\chi_{2j}\left(\sqrt{\frac{\sigma}{\rho}}\right)$ gives the character of the spin j representation of $SU(2)$. Use the above information to write down all possible contributions in the Coulomb and Schur limits of the index.

5. Write the $\mathcal{N}=2$ index that was defined in the lecture as a path integral on $S^3\times S^1$ with periodic boundary conditions for all fields.

6. The contribution of the $\mathcal{N}=4$ vector multiplet to the "$\mathcal{N}=4$ index" with some choice of fugacities is

$$i^{\text{vec mult}}_{\mathcal{N}=4} = \frac{t^2\left(v+\frac{1}{w}+\frac{w}{v}\right) - t^3\left(y+\frac{1}{y}\right) - t^4\left(w+\frac{1}{v}+\frac{v}{w}\right) + 2t^6}{(1-t^3y)\left(1-\frac{t^3}{y}\right)} .$$

Carry out the gauge integration at large N to show that the index for $U(N)$ 4D $\mathcal{N}=4$ SYM reproduces the index over free IIB supergravitons on $AdS_5\times S^5$, given by

$$\mathcal{I}^{grav} = \prod_{n=1}^{\infty} \frac{(1-t^{3n}/y^n)\,(1-t^{3n}y^n)}{(1-t^{2n}/w^n)\,(1-v^n t^{2n})\,(1-t^{2n}w^n/v^n)} .$$

You can use that $\chi_{adj}(U) = \mathrm{Tr}(U^\dagger)\mathrm{Tr}(U)$ and $[dU] = \frac{1}{(2\pi)^N N!}\prod_i d\lambda_i \prod_{i<j} 4\sin^2\left(\frac{\beta(\lambda_i-\lambda_j)}{2}\right)$, where $U\equiv e^{i\beta\alpha}$ and λ_i $(i=1,\cdots,N)$ the eigenvalues of α.