"A crash course on the Superconformal Index"

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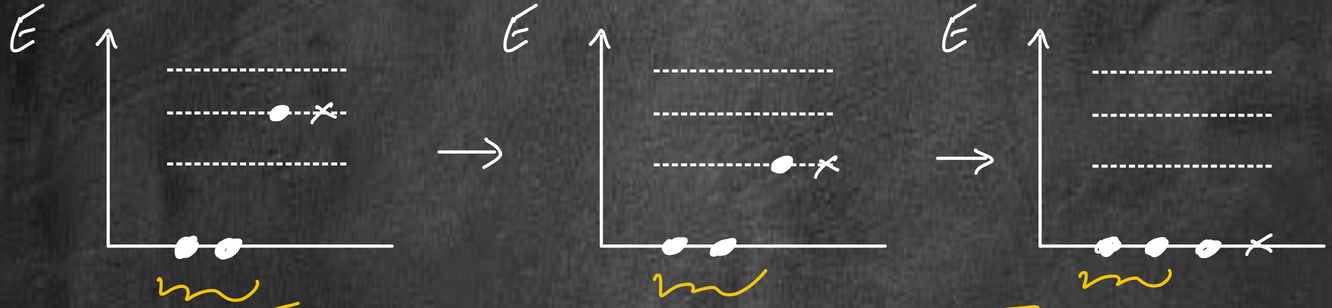
Goal: Introduce & highlight properties of the SCI (broad picture
not comprehensive)

Outline:

- Motivation
- Some rep theory on structure of UIRs
- Definition for 4D $\mathcal{N}=2$ SCFTs
- Limits of 4D $\mathcal{N}=2$ index with additional susy
- Applications / Connections

Motivation

→ In susy Quantum Mechanics Witten Index counts susy ground states that do not pair up under continuous deformations of the theory (susy preserving)



Witten index counts this

[Witten '82]

⇝ Innocent-looking modification of the partition function:

$$I = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H}$$

$\mathcal{H}$: space of states

F : Fermion number operator

H : Hamiltonian

⇝ For SUSY QM :

$$[Q, H] = 0$$

$$[\bar{Q}, H] = 0$$

$$\{Q, \bar{Q}\} = H$$

⇝ Consider $|\psi\rangle$ such that $H|\psi\rangle = E|\psi\rangle$

$$\bar{Q} H |\psi\rangle = H \bar{Q} |\psi\rangle = E(\bar{Q}|\psi\rangle) \quad \text{for } E \neq 0$$

→ Bosonic & fermionic states with $E \neq 0$ linked by susy

⇒ Because of $(-1)^F$ they cancel out in the index

⇒ The Witten index is a robust quantity

(Can calculate with interactions turned off and then it will still be accurate when they are turned on)

→ Less information but more control than partition function

→ For the Superconformal index: Poincaré superconformal

Roughly replace : $H = \{Q, \bar{Q}\} \longrightarrow \{Q, S\}$

susy vacua $\longrightarrow$ "Short UIRs of the SCA"

Some rep. theory on structure of UIRs

⇝ The complete list of SCAs is known. For field theories:

$D=3 \qquad osp(N|4) \supset so(N) \oplus so(2,3)\,, \; N \leq 8$

$$D=4 \quad \begin{cases} su(2,2|N) \supset so(2,4) \oplus u(N) & , N=1,2,3 \\ psu(2,2|4) \supset so(2,4) \oplus su(4) \end{cases}$$

$D=5 \qquad osp(2,5|2) \supset so(2,5) \oplus usp(2)$

$D=6 \qquad osp(2,6|2k) \supset so(2,6) \oplus usp(2k)\,, \; k=1,2$

[Kac '77, Nahm '78]

⇝ All states in an SCFT fall under UIRs of the corresponding SCA ⇒ we can classify and build these

⇝ A SCA contains the following generators:

Lorentz : M

R-symmetry : R

Dilatation : D

Momentum : P

Special conformal : K

bosonic

Poincaré susy : Q

Superconformal susy : S

fermionic

$\rightsquigarrow$ Representations are built as follows: [Dobrev-Petkova '85
Minwalla '97]

- Start with a superconformal primary $|\psi\rangle$

superconformal primaries $S|\psi\rangle = 0\,,\; K|\psi\rangle = 0$

conformal primaries $K|\psi'\rangle = 0$

- Identify maximal, compact, bosonic subalgebra

E.g. for 4D $\mathcal{N}=2$:

$$su(2,2|2) \supset so(2,4) \oplus u(2) \supset \underbrace{so(2)}_{\Delta} \oplus \underbrace{so(4)}_{M} \oplus \underbrace{u(2)}_{R}$$

- Superconformal primaries are in 1-1 correspondence with highest weight states of this compact subalgebra: $|\psi\rangle \equiv |\Delta; M; R\rangle^{h.w.}$

⇒ Superconformal descendants obtained by acting with Q's

⇒ Conformal descendants obtained by acting with P's

⇝ A basis for the representation space of the SCA is given by:

$$\prod Q^{n} \prod P^{\hat{n}} |\Delta; \mu; R\rangle^{h.w.}$$ → Generic, long multiplet

Note: Finite number of superconformal descendants but infinite number of conformal descendants

⇝ However $\exists$ examples for which $\prod Q |\Delta; \mu; R\rangle^{h.w.} = 0$ → short multiplet

⇝ How to classify these short multiplets?

⇒ Systematic approach using Euclideanised version of SCA

⇝ We will be working w/ SCFTs in flat space, Euclidean signature and in radial quantisation

⇝ In Euclidean version the generators are not hermitian:

$$P^{\dagger} = K \quad \text{and} \quad Q^{\dagger} = S \qquad [K, P] \propto M + D \qquad \{Q, S\} \propto M + D + R$$

⇝ Unitarity needs to be imposed ∀ states. Null states are removed and multiplet is short. Leads to shortening conditions

E.g.
$$\| Q | \Delta; M; R \rangle^{h.w} \|^2 = {}^{h.w.}\langle \Delta; M; R | S Q | \Delta; M; R \rangle^{h.w} \rightarrow M + \Delta + R = 0$$
$$= {}^{h.w}\langle \Delta; M; R | M + D + R | \Delta; M; R \rangle^{h.w}$$

Comments:

- This is sufficient to completely classify all UIRs for SCAs relevant for field theory
- One can also explicitly construct them:

  4D $\mathcal{N}=2$, $\mathcal{N}=4$ [Dolan-Osborn '02]

  5D, 6D [Buican-Hayling-C.P. '16]

  All 3D-6D [Córdova, Dumitrescu, Intriligator '16]
- Simple short multiplets are vectors, hypers, currents etc.
- SCA constrains what can happen but does not say what will happen

Definition for 4D $\mathcal{N}=2$ SCFTs

↝ We can finally define the superconformal index:

[Romelsberger '05
Kinney-Maldacena-Minwalla-Raju '05]

- Pick an SCFT ⇒ SCA
- Pick one supercharge and construct

$$\mathcal{I} := \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta \{Q, S\}}$$

↪ Space of local quantum operators

- Bosonic/fermionic states for which $\{Q, S\}|\psi\rangle \neq 0$ pairwise cancel. Those with $\{Q, S\}|\psi'\rangle = 0$ belong to short multiplets of the SCA

Comments:

- Like for Witten index sometimes under continuous deformations short multiplets recombine into long ones

  These are known via "recombination rules"

  $\Rightarrow$ The superconformal index does not receive contribution from such combinations

  E.g. $\mathcal{L} \xrightarrow{\epsilon \to 0} \mathcal{A} \oplus \mathcal{B}$ then $\mathcal{I}(\mathcal{A}) = -\mathcal{I}(\mathcal{B})$ even when $\epsilon = 0$

  $\Rightarrow$ The index is a number and cannot change continuously

- Not all states in a short multiplet contribute to the index

- Representations of SCA are infinite-dimensional so naive index diverges $\Rightarrow$ Refine the index

→ For the maximal refinement we can use maximal set of SCA generators (or their linear combinations) that commute with Q and S commutant of SCA

E.g. say that the commutant is c_1, c_2, c_3 then the refined index will be given by

$$\mathcal{I} := \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta\{Q,S\}} p^{c_1} q^{c_2} t^{c_3}$$

fugacities

$$\begin{cases} p = \exp(-\beta u_p) \\ q = \exp(-\beta u_q) \\ t = \exp(-\beta u_t) \end{cases}$$

$u_p, u_q, u_t \to$ chemical potentials

Note: One can perform additional refinements w/ symmetries that are not part of the SCA

→ Focus on 4D $\mathcal{N}=2$ for concreteness:

[Gadde-Rastelli-Razamat-Yan '11]

SCA is $SU(2,2|2) \supset SO(2) \oplus SO(4) \oplus U(2)$

$\simeq SO(2) \oplus SU(2)_1 \oplus SU(2)_2 \oplus SU(2)_R \oplus U(1)_r$

Quantum numbers: Δ j_1 j_2 R r

→ There are also $4\ P_\mu$, $4\ K_\mu$, $4\ Q_{I\alpha}$, $4\ \tilde{Q}_{I\dot{\alpha}}$, $4\ S^{I\alpha}$, $4\ \tilde{S}^{I\dot{\alpha}}$

$$\{\tilde{S}^{I\dot{\alpha}}, \tilde{Q}_{J\dot{\beta}}\} = \delta^I_J \left(M^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} D\right) - \delta^{\dot{\alpha}}_{\dot{\beta}} R^I{}_J \qquad (I=1,2,\ \alpha=\pm,\ \dot{\alpha}=\dot{\pm})$$

→ Construct the index using $\tilde{Q}_{1\dot{-}}$ and $(\tilde{Q}_{1\dot{-}})^\dagger = \tilde{S}^{1\dot{-}}$

These commute with the maximal set:

$$\delta_{1+} := \Delta + 2j_1 - 2R - r, \quad \delta_{2+} := \Delta - 2j_1 - 2R - r, \quad \tilde{\delta}_{2\dot{+}} := \Delta + 2j_2 + 2R + r$$

$$\delta_{1+} = 2\{Q_{1+}, S^{1+}\}, \quad \delta_{2+} = \{Q_{2+}, S^{2+}\}, \quad \tilde{\delta}_{2\dot{+}} = \{\tilde{Q}_{2\dot{+}}, \tilde{S}^{2\dot{+}}\}$$

⇝ So we define the refined index via:

$$\mathcal{I}(p,q,t) := \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-2\beta \{\tilde{Q}_{1\dot{-}}, \tilde{S}^{1\dot{-}}\}} p^{\frac{1}{2}\delta_{1-}} q^{\frac{1}{2}\delta_{1+}} t^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}}$$

$F = 2j_1$

Note 1: We could have defined $\mathcal{I}$ w.r.t. any other supercharge. The commuting subalgebra is different but the final answer for $\mathcal{I}$ remains the same

Note 2: The indices for different SCAs (but for the same theory) are different. E.g. the "$\mathcal{N}=4$" index for $\mathcal{N}=4$ SYM is different from the "$\mathcal{N}=2$" index for $\mathcal{N}=4$ SYM

$\rightsquigarrow$ This is all great but how do we calculate ?

$\Rightarrow$ One could do this if one knew $\mathcal{H}$

$\rightsquigarrow$ for theories with a marginal deformation $\exists$ a great simplification: Evaluate the index in free limit and we are guaranteed it doesn't change at any other value of the coupling!

Caveat: There could be a discrete jump as one goes from zero to small but nonzero coupling

⇝ For Lagrangian theories things are now looking better:

- $\mathcal{H}$ comprises the set of all local operators that one can construct from free fields ← fields appearing in Lagrangian
- Each field is a letter while the composites they build are words
- Index over letters is known as the single-letter index, $i(t)$

- Equations of motion obeyed by free fields are relations between operators and contribute to the index with opposite sign

- The index over all words can then be obtained via the plethystic exponential:

$$\mathcal{I}(t) = \text{P.E.}[i(t)] := \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} i(t^n)\right]$$

- Finally, in a gauge theory we want to consider gauge-invariant operators, i.e. gauge singlets.

- Append a group character in the appropriate representation R_j for the single-letter index

  i.e. $\chi_{R_j}(U)$ with U an element of the gauge group

- Characters obey the orthogonality property:

$$\int [dU]\, \chi^{\dagger}_{R}(U)\, \chi_{R'}(U) = \delta_{RR'} \quad \text{and also} \quad \chi_{R_1 \otimes R_2} = \chi_{R_1} \cdot \chi_{R_2}$$

↑ Unit normalised Haar measure

$\rightsquigarrow$ Therefore : $\int [DU] \, \chi^{*}_{R}(U) \prod_{i=1}^{n} \chi_{R_i}(U) = n_R$

number of R reps in $R_1 \otimes R_2 \otimes \dots \otimes R_n$

and to keep track of the number of gauge singlets

employ the above for $\chi^{*}_{R}(U) = 1$

$\Rightarrow$ We have arrived at the final expression of index for gauge theories with a marginal coupling :

$$\mathcal{I}(t) = \int [DU] \, \text{P.E.}[\, i(t, U)\,] = \int [DU] \, \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \, i(t^n, U^n) \right]$$

↳ E.g. for the "$\mathcal{N}=2$" index of $\mathcal{N}=4$ SYM

$$I^{\mathcal{N}=4} = \int [dU]\ \text{P.E.}\Big[\big(\underbrace{i_V(\sigma,\rho,\tau)}_{\mathcal{N}=2\text{ vector multiplet}} + \overbrace{i_H(\sigma,\rho,\tau)}^{\mathcal{N}=2\text{ hypermultiplet}}\big)\,\chi_{\mathrm{adj}}(U)\Big]$$

and so on for other SCFTs

Note: In practice difficult to perform gauge integral exactly. Either low rank, or large-N, or expand order-by order in fugacities

Limits of 4D $\mathcal{N}=2$ index with additional SUSY

⇝ There are interesting fugacity limits one can consider

→ The $\mathcal{N}=2$ index was given by:

$$\mathcal{I}(\rho,\sigma,\tau) := \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-2\beta\{\tilde{Q}_{1\dot{-}}, \tilde{S}^{1\dot{-}}\}} \rho^{\frac{1}{2}\delta_{1-}} \sigma^{\frac{1}{2}\delta_{1+}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}}$$

$$2\{\tilde{Q}_{1\dot{-}}, \tilde{S}^{1\dot{-}}\} =: \tilde{\delta}_{1\dot{-}} = \Delta - 2j_2 - 2R + r$$

⇝ Multiplets that contribute are annihilated by $\tilde{Q}_{1\dot{-}}$

⇒ They are 1/8 - BPS

$\leadsto$ Macdonald limit : $\quad \sigma \to 0\,,\ \rho, \tau \to$ fixed

Only states obeying both $\left\{ \begin{array}{l} \tilde{\delta}_{1\dot{-}} = 0 \\ \delta_{1+} = 0 \end{array} \right\}$ contribute

$$\mathcal{I}_{\mathcal{H}} = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta(\Delta + 2j_1 - 2R - r)} \rho^{\frac{1}{2}(\Delta - 2j_1 - 2R - r)} \tau^{\frac{1}{2}(\Delta + 2R + 2j_2 + r)}$$

$\Rightarrow$ These multiplets are annihilated by both $\tilde{Q}_{1\dot{-}}, Q_{1+}$ and are 1/4 BPS

→ Hall-Littlewood limit : $\sigma \to 0,\ \rho \to 0,\ \tau \to$ fixed

$$\mathcal{I}_{HL} = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta(\Delta + 2j_1 - 2R - r)} \tau^{\frac{1}{2}(\Delta - R)}$$

⇒ The multiplets contributing are annihilated by

Q_{1+}, Q_{1-}, $\tilde{Q}_{1\dot{-}}$ and obey:

$$j_1 = 0,\quad j_2 = r,\quad \Delta = 2R + r$$

For Lagrangian, linear quiver theories the HL index coincides with the Higgs-branch Hilbert series

$\rightsquigarrow$ Schur limit: $\rho = \tau$, $\sigma \to$ fixed

One finds that:

$$\mathcal{I}_S = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta(\Delta - 2j_2 - 2R + r)} \sigma^{\frac{1}{2}(\Delta + 2j_1 - 2R - r)} \rho^{\Delta - j_1 + j_2}$$

$\Rightarrow$ It turns out that the combinations of charges commutes not only with $\tilde{Q}_{1\dot{-}}$ but also Q_{1+}

$\mathcal{I}_S$ is independent of both β and σ and

$$\mathcal{I}_S = \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta(\Delta - 2j_2 - 2R + r)} \sigma^{\frac{1}{2}(\Delta + 2j_1 - 2R - r)} \rho^{2(\Delta - R)}$$

⇝ Coulomb-branch limit : $\tau \to 0$, $\rho, \sigma \to$ fixed

$$\mathcal{I}_C = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta(E-2j_2-2R+r)} \sigma^{\frac{1}{2}(\Delta+2j_1-2R-r)} \rho^{\frac{1}{2}(\Delta-2j_1-2R-r)}$$

⇒ Multiplets contributing are annihilated by both $\tilde{Q}_{1\dot{-}}, \tilde{Q}_{2\dot{+}}$

Hypermultiplets do not contribute in this limit.

⇝ Similar limits can be found for other indices

⇝ Many instances where only such limits can be calculated because:

- We have knowledge of the spectrum of contributing ops.
- For technical reasons calculations are tractable

Applications / Connections

- Index is invariant under continuous deformations such as marginal couplings
  - ⇒ Check for strong-weak dualities (AdS/CFT) [Kinney-Maldacena-Minwalla-Raju '05]
- Similarly for theories related by RG flows, as long as the symmetries involved in its definition survive
  - ⇒ Check of IR dualities in 4D [Dolan-Osborn '08]
  - ⇒ Dualities in 3D

→ Some of these calculations are done using the following connection:

local operators in $\mathbb{R}^D$ flat-space, Euclideanised CFT in radial quantisation $\xleftrightarrow[1-1]{\text{Operator-State map}}$ States in the theory on $S^{D-1} \times S^1_\beta$

S^{D-1} ↑ $SO(D)$, S^1_β ↑ $SO(2)$

$$\Rightarrow \quad \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} \longleftrightarrow \int [\mathcal{D}x]\, e^{-S_E[x]}$$

$$\Rightarrow \quad \mathrm{Tr}_{\mathcal{H}}\, (-1)^f e^{-\beta H} \longleftrightarrow \int \mathcal{D}[x, \psi]\, e^{-S_E[x,\psi]}$$

with periodic b.c.'s for the fermions

$$\Rightarrow \quad \mathrm{Tr}_{\mathcal{H}}\, (-1)^f e^{-\beta\{Q,S\}} e^{-\beta \mu_i M_i} \longleftrightarrow \quad \gg \quad \text{with} \quad H \to \Delta + \alpha(\mu_i, M_i)$$

⇒ For Lagrangian theories these P.I.s can be evaluated exactly using susy localisation

Note: There are many susy theories on $S^3 \times S^1_\beta$ which are not conformal. The P.I. can also be evaluated in those cases using localisation but it does not correspond to the index.

$\rightsquigarrow$ $\exists$ an isomorphism between the Schur Sector of 4D $\mathcal{N}=2$ SCFTs (including non-Lagrangian) and a 2D chiral algebra

[Beem-Lemos-Liendo, Peelaers-Rastelli-Van Rees '13]

Schur index in 4D

$\Updownarrow$

Vacuum character in 2D

$\Rightarrow$ Use this to evaluate many indices for non-Lagrangian theories

$\Rightarrow$ Also other options using TQFTs for class-S ...