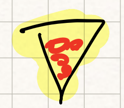
LonTI lectures: on SYK and the emergence of spacetime

Lecturer: Damián Galante (KCL)

Dates & Time: Mondays 21, 28, 7, 14 @ 10:30 hs.

(pizza afterwards) 

* Round of names

* **Format**: Informal, please ask many questions
  BUT bear in mind this is intended for 1st year PhD students.

Plan for lectures:

1. Intro & motivation ← TODAY!!
2. JT-gravity & $nAdS_2$ spacetimes

---

3. The SYK model
4. thermodynamics & correlators in the IR: the emergence of spacetime

Maybe: extra Mathematica lecture w/ Kolya Gromov

MAIN GOAL: Show that the thermodynamics & correlation functions of the SYK model in the IR are those of a "black-hole" in 2d spacetime.

* computation of a particular 4pt function: the out-of-time-ordered.

What I'll assume you know:

Basic GR: ∇_μ, $\Gamma^\alpha{}_{\mu\nu}$, $R^\alpha{}_{\mu\nu\beta}$, geodesics, etc.
Killing symmetries, Einstein action
Some idea of BH-thermo.

Basic QFT: Green's functions, Feynman diagrams
Some idea of what conformal symmetry is.
Path integral formalism

What I don't assume you know: AdS spacetimes, string theory, SUSY, ...
AdS/CFT

Bibliography:

* Lecture notes by Gábor Sarosi
  arXiv: 1711.08482
  "AdS_2 holography and the SYK model"

* Maldacena, Stanford & Yang
  "Conf. symmetry & its breaking in two-dimensional nearly Anti-de Sitter space"
  arXiv: 1606.01857

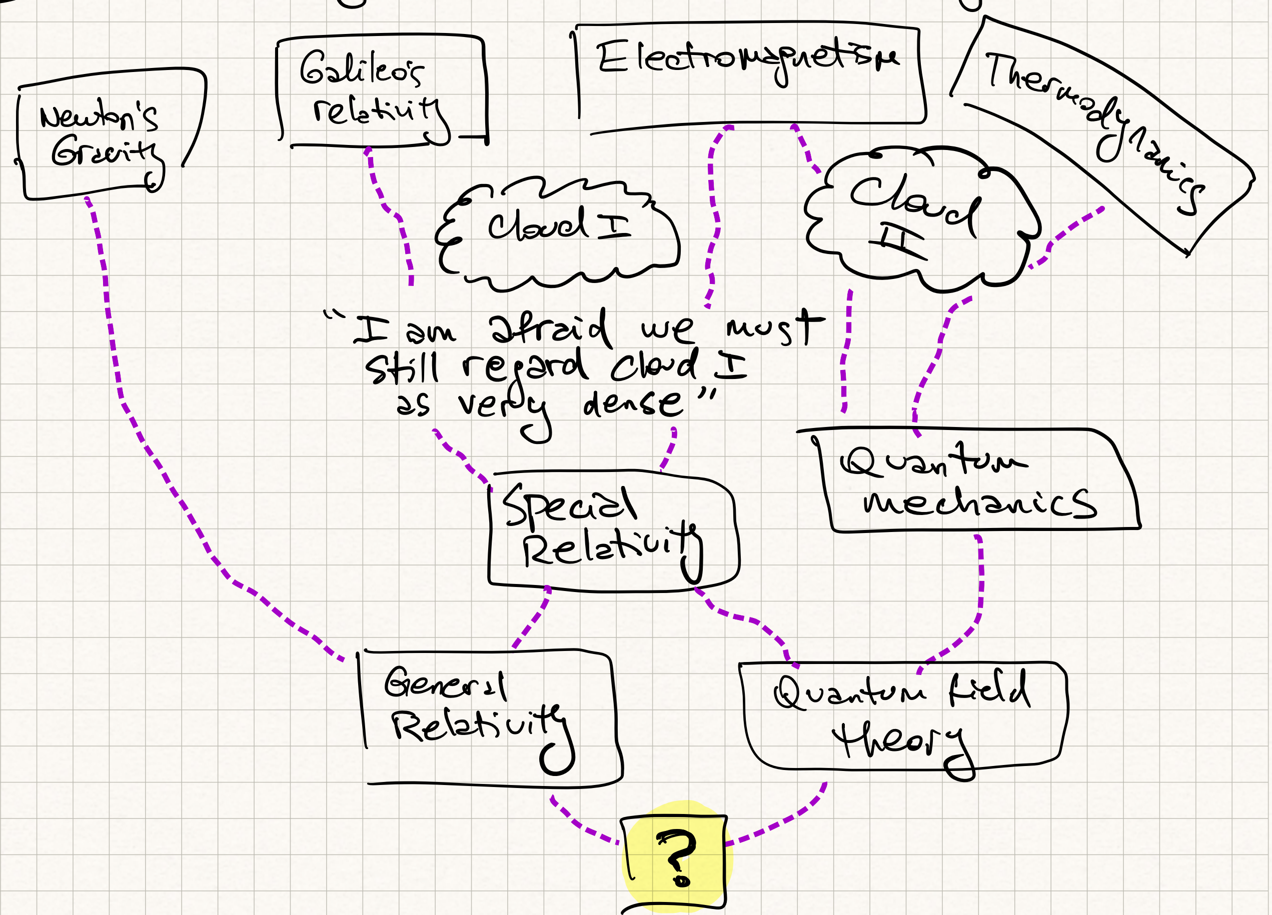
* Maldacena & Stanford
  "Comments on the Sachdev-Ye-Kitaev model"
  arXiv: 1604.07818

- Others: Videos by Stanford, Rosenhaus, Trunin, etc.

Lecture 1 : Historical overview & motivation

- We are at the RI, so I cannot start these lectures without going through this:

1900: 4 building blocks of fundamental physics



- But we still couldn't find a quantum theory of gravity (maybe string theory??)
- Maybe there's an alternative way of thinking about the problem $\Rightarrow$ what if gravity is not fundamental but emergent from certain quantum systems?

* For now let's assume this is the case. Then, what type of quantum system this should be??

↳ Here is where the idea of holography starts forming.

⇒ '70's, $\boxed{S_{BH} = \frac{A}{4} \frac{k_B c^3}{\hbar G_N}}$ ⇒ entropy scales like area??

(From now on $k_B = c^3 = \hbar = 1$)

⇒ 't Hooft, Susskind ⇒ $\boxed{\text{Gravity in } d+1}$ ⟺ $\boxed{\text{Quantum system in } d}$

⇒ Maldacena in '98 ⇒ $\boxed{\text{Gravity in } AdS_{d+1}}$ ⟺ $\boxed{\text{CFT in } d \text{ dimensions}}$

But $d > 1$. Already in '98 Maldacena, Michelson, Strominger pointed out about "the enigmatic case of $d=1$".

⇒ Two reasons to study AdS_2:

① On the quantum side, the theory should be a "0+1" dimensional theory ⇒ $\boxed{\text{IT HAS NO SPATIAL DIMENSIONS}}$

⇒ everything happens at a single point that evolves in time.

(*) Best place to actually see the emergence of spacetime (Borges, "the Aleph")

⊛ But it can also be "the easiest": the dual theory will have finite N degrees of freedom as opposed to the QFT versions in higher dimensions. (might even be computationally tractable).

② Practical reason: AdS_2 appears universally as the near-horizon geometry of (nearly) extremal black holes in higher dimensions.

But still, it took more than 15 years to get to the story I want to teach you in these lectures:

⇒ '15 $\boxed{\text{Gravity in } nAdS_2}$ ⟷ $\boxed{\text{QM w/disorder SYK}}$

SY: '92

K: '15 no paper.

$\boxed{\text{Quite new: there's still lots to understand}}$

Plan for today:

* Show that AdS_2 appears in near-horizon geometries

* Discuss some properties of AdS_2 space

* Show how JT gravity appears in the game.

(1) The Reissner-Nordstrom black hole.

Consider the following theory in 4d:

$$S_L = \frac{1}{16\pi G_N}\int d^4x \sqrt{-g}\, R - \frac{1}{4}\int d^4x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} + S_{matter}$$

(classical GR coupled to electromagnetism + some matter)

This theory admits the following solution:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$(S_{matter} = 0)$

$$A = \frac{Q}{r}\,dt$$

This is known as the RN-BH

Of course, one can first ask what is the position of the horizon: [Penrose diagram]

$$f(r_h) = 0 \Rightarrow \text{Quadratic equation} \Rightarrow r_{h\pm} = M \pm \sqrt{M^2 - Q^2}$$

(two horizons)

$M \geq Q$

If $Q = 0 \Rightarrow r_h = 2M$

According to BH formula $\Rightarrow S_{BH} = \dfrac{\pi r_{h+}^2}{G_N}$

And we can also compute the temperature of this BH:

$$\boxed{T_H = \frac{1}{4\pi} f'(r_h) = \frac{r_{h+} - r_{h-}}{4\pi r_{h+}^2}}$$

Note now that something interesting happens when

$M \sim Q \Rightarrow r_{h+} \sim r_{h-} \Rightarrow T_H \cong 0$. But still

$r_{h+} \neq 0$ & $S_{BH} \neq 0$!!

this is called the extremal limit. [Penrose diagram]

(*) Let's see how the geometry looks close to the horizon. We will define a dimensionless parameter

$$\boxed{\varepsilon \equiv \frac{M-Q}{Q}} \quad \text{at fixed } Q.$$

For small ε, we have $r_h \cong Q(1+\sqrt{2\varepsilon}+\cdots)$

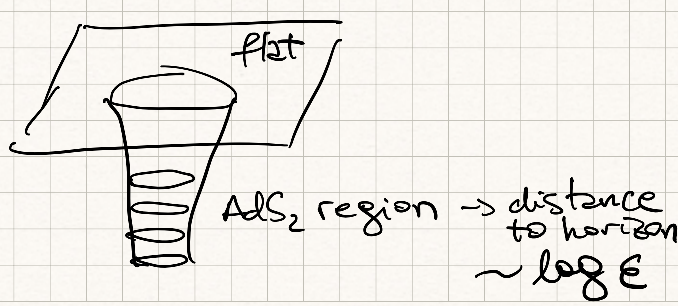
The near horizon geometry can be obtained by expanding close to $r_h \rightarrow r = Q(1+\sqrt{2\varepsilon}\,\rho)$

$$t = \frac{Q}{\sqrt{2\varepsilon}}\tau$$

$$\Rightarrow \boxed{\frac{ds^2}{Q^2} = \underbrace{-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}} + \underbrace{d\Omega_2^2}}$$

$$ds^2_{ext} = AdS_2 \times S^2$$

[Penrose diagram]



If we look at the correction to the temperature and entropy to leading order in ε we obtain:

$$T_H = \frac{\sqrt{\varepsilon}}{\sqrt{2\pi}\,Q} + \mathcal{O}(\varepsilon)$$

$$S_{BH} = \frac{\pi Q^2}{G_N}\left(1 + 2\sqrt{2\varepsilon} + \cdots\right)$$

$$= \frac{\pi Q^2}{G_N}\left(1 + 4\pi Q\, T_H + \cdots\right)$$

↳ linear in temperature!!

② A few words about AdS_2

AdS_2 is the maximally symmetric solution with negative curvature.

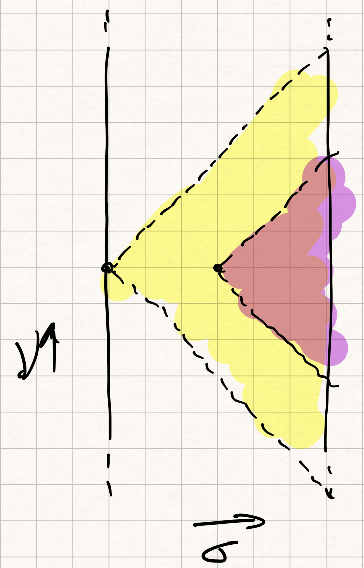
* AdS_2 (in contrast with AdS_{d+1}) has 2 boundaries
* the Penrose diagram is an infinite strip

In global coordinates the metric is given by:

$$\boxed{ds^2 = \frac{-dv^2 + d\sigma^2}{\sin^2\sigma}}$$

It becomes infinitely large at the boundaries where $\sigma = 0, \pi$.

And global time $v \in [-\infty, \infty]$



Other coordinates cover different regions of the Penrose diagram.

One typical one is the Poincaré patch where $ds^2 = \dfrac{-dt^2 + dz^2}{z^2}$

with $z \geq 0$

The Rindler or Black hole patch is given by

$$ds^2 = d\rho^2 - \sinh^2\rho \, d\hat{\tau} = -(r^2-1)\,d\tau^2 + \frac{dr^2}{(r^2-1)}$$

2.1 Symmetries:

For this we will use Poincaré coordinates

$$ds^2 = \frac{-dt^2 + dz^2}{z^2}$$

* A maximally symmetric spacetime in d dimensions has $\dfrac{d(d+1)}{2}$ $\Rightarrow$ AdS_2 has 3 killing vectors

HW: Verify that $\xi_H = \partial_t$, $\xi_D = t\partial_t + z\partial_z$

$$\xi_K = \left(\frac{z^2}{2} + \frac{t^2}{2}\right)\partial_t + tz\partial_z$$

Under the Lie bracket they generate the $SL(2,\mathbb{R})$ algebra

$$[\xi_H, \xi_D] = -\xi_H, \quad [\xi_D, \xi_K] = -\xi_K$$

$$[\xi_H, \xi_K] = -2\xi_D$$

Another way of seeing very explicitly how the $SL(2,\mathbb{R})$ symmetry appears is by noting that the Poincaré Patch is invariant under:

$$z \pm t \to \frac{a(z \pm t) + b}{c(z \pm t) + d}, \quad ad - bc = 1$$

(Möbius transformations)

* A third way: the hyperboloid in Embedding space.
$-Y_{-1}^2 - Y_0^2 + Y_1^2 = -1$, embedded in $\mathbb{R}^{2,1}$ $(ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2)$

2.2 Euclidean AdS

Many times it will be useful to do computations in Euclidean space. We take $t \to it_E$ and we make t_E periodic $\Rightarrow$ generates the famous hyperbolic disk

$$ds^2 = \frac{dt^2 + dz^2}{z^2} = \frac{d\rho^2 + d\tau^2}{\sinh^2\rho} = d\rho^2 + \sinh^2\rho \, d\tau^2.$$

[Escher drawing]

2.3 What's wrong with AdS_2 ?? (comment on the two-boundaries)

- Einstein gravity in 2d is topological. Consider

$$S_{EH} = \underbrace{\left[\int_M d^2x \sqrt{g} R + 2 \int_{\partial M} K \right]}_{2\pi\chi \text{ for any metric}}$$

* Another way of saying this is that the

E.O.M. give $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$ (as always)

but in 2d $R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$ for any metric

so the E.O.M vanishes identically.

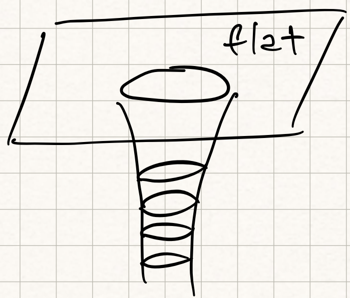
Yes, AdS_2 is a solution but so are all other metrics.

* Moreover, we might want to couple this theory to matter ⇒ it looks like we can only do $T^{matter}_{\mu\nu} = 0$ ⇒ "Gravity is over-constraint in 2d".

* It is only a theory of ground states but it cannot account for excitations.

③ The appearance of JT Gravity

So what do we do? we will take inspiration from the behaviour of higher dimensional black holes near extremality.



→ $AdS_2 \times S^2$ ⇝ so the radius is almost constant but what if we allow it to vary slowly as we move in the AdS_2??

$$\Rightarrow \frac{\text{Area}}{16\pi G_N} = \Phi_0 + \phi \,, \text{ where } \Phi_0 \gg \phi$$

The calculation is a bit lengthy and I don't want to get into the details but basically what we need to do is:

* Start in 4d with:

$$S_L = \frac{1}{16\pi G_N}\int d^4x \sqrt{-g}\, R - \frac{1}{4}\int \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}$$

we assume a spherically symmetric ansatz for the metric and field strength

$$ds^2 = h_{ij}\, dx^i dx^j + e^{2\psi} d\Omega_2^2$$

$$F = Q \sin\theta \; d\phi \wedge d\theta$$

Now we plug this in the action and do the angular integrals. the result looks like

$$S_L \sim \frac{1}{G_N}\int d^2x \sqrt{h}\left[e^{2\psi}\left(R_h + 2(\partial\psi)^2\right) + 2 - \frac{G_N}{2} e^{-2\psi} Q^2\right]$$

Next we call $e^{\psi} \equiv \Phi$. In the extremal case $\Phi^2 = \Phi_0$ constant, but we want to slightly deviate from that

$\Rightarrow \Phi^2 = \Phi_0 + \phi$ with $\Phi_0 \gg \phi$

$$\Rightarrow \boxed{S_{JT} = \frac{\Phi_0}{16\pi G_N}\int_M d^2x \sqrt{-h}\, R_h + \frac{1}{16\pi G_N}\int_M d^2x \sqrt{h}\, \phi (R_h + 2) \quad \left(+2\int_{\partial M} K\right) \quad \left(+2\int_{\partial M} \phi K\right) + \dots}$$

⇒ This is what we call JT gravity theory. It has not only the metric but also a dilaton field (that can be seen as the size of the transverse space in higher dimensions).

The existence of this running dilaton will break some of the symmetries of AdS_2 and will make the theory more interesting.

⇒ Next lecture we will study this theory in depth and understand its basic properties.

* HOMEWORK EXERCISES

① Hawking temperature of black holes. Assume you have a metric of the form:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2$$ and that $f(r)$ has a simple zero at $f(r_h) = 0$.

a) Take $t \to -i\tau_E$

b) Expand the metric close to $r \sim r_h$

c) Make the change of coordinates

$$\rho = 2\sqrt{\frac{r-r_h}{f'(r_h)}}\ ; \quad \theta = \frac{t_E}{2} f'(r_h)$$

to realise the metric looks like the planar metric

d) Impose the right periodicity to avoid the conical singularity. You should find that if $t_E \sim t_E + \beta \Rightarrow \beta = \frac{1}{T} = \frac{4\pi}{f'(r_h)}$.

(2) Repeat this procedure for the near-extremal Reissner-Nordstrom black hole to verify the reseults in the notes.
Also compute the Area of the horizon and see that it is proportional to r_{h+}^2.

(3) Verify that the Killing vectors of AdS_2 satisfy the $SL(2,\mathbb{R})$ algebra.

(4) Show that the Poincaré patch metric is invariant under $z \pm t \to \frac{a(z\pm t)+b}{c(z\pm t)+d}$,

with $ad-bc=1$. Can you check which parts of the Penrose diagram are covered by these transformations??

(5) Verify the claim that in 2 spacetime dimensions, $R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$ for any metric $g_{\mu\nu}$.