LonTI lectures: on SYK and the emergence of spacetime

Lecture 2: on JT gravity and $nAdS_2$

* Nikolay Gromov agreed to record a lecture on SYK model using Mathematica.

Very brief recap:
* Holography in 2d is interesting by its own.
* Near-extremal black holes in higher dimensions have a near-horizon geometry of $AdS_2 \times S^2$.
* The effective action in 2d is that of Jackiw-Teitelboim gravity (JT)

Linear in T specific heat!!

TODAY:
* We will study JT gravity theory.
* We will show it reduces to a boundary theory.
* We will also show its thermodynamics reproduce the linear-in-T specific heat.

2.1 JT gravity action: (we will work in Euclidean signature)

$$S_{JT} = -\frac{\Phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x \sqrt{g}\, R + 2\int_{\partial\mathcal{M}} \sqrt{h}\, K\right]$$

$$-\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x \sqrt{g}\, \phi (R+2) + \int_{\partial \mathcal{M}} \sqrt{h}\, \phi K\right]$$

A few comments about this action:

1. the obvious one: it is 2-dimensional. It has a metric $g_{\mu\nu}$ and a scalar field ϕ, that we will call the dilaton.
2. The first term is topological (proportional to the Euler characteristic of $\mathcal{M}$).
3. Inspired in higher-d we want to study
$$\Phi = \phi_0 + \phi \, , \quad \text{where } \phi_0 \gg \phi$$
(JT is the leading correction to pure AdS_2).
4. We will study the theory in Euclidean signature (I might make a few comments about the Lorentzian theory at the end) → why?
   * Thermodynamics
   * Correlators easier to compute.
   * Lorentzian correlators by analytic continuation.

— o —

The problem is not well-defined without imposing boundary conditions (Dirichlet):

$$\phi|_{bdy} = \phi_b = \frac{\tilde{\phi}}{\varepsilon} \; ; \quad h|_{bdy} = \frac{\tilde{h}^2}{\varepsilon^2}$$

($\tilde{\phi}$ → dimensionful constant!!)

* Now let's look at the equations of motion for the metric and the dilaton: (III)

$$\frac{\delta}{\delta\phi}: \quad R+2=0 \Rightarrow R=-2$$

→ huge simplification in 2d ⇓ the metric is locally AdS_2.

(this won't change even if we add matter, given it does not interact with ϕ)

$$\frac{\delta}{\delta h}: \quad \underbrace{\frac{1}{8\pi G}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi + g_{\mu\nu}\phi\right)}_{T^{\phi}_{\mu\nu}} = 0$$

$$T^{\phi}_{\mu\nu}=0.$$

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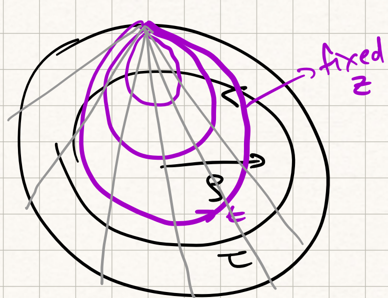
* Comment on path integral formulation:
(we are really integrating out the dilaton)

$$Z = \int Dg D\phi \, e^{-S[g,\phi]}$$

(If you properly chose the contour of integration)

$$\int Dg \int D\phi \, e^{-i\int\phi(R+2)} = \int Dg\, \delta(R+2)$$

⊛ $R=-2 \Rightarrow$ geometry is Euclidean AdS_2:



We will use two sets of coordinates to describe $EAdS_2$. Both cover the full disk:

- $\tau_{Rindler} \to i\,\tau_{Rindler}$

$$ds^2 = d\rho^2 + \sinh^2\rho \, d\tau^2$$

- $t_{Poincaré} \to i\, t_{euclidean}$

$$ds^2 = \frac{1}{z^2}(dt^2 + dz^2)$$

We will see in what sense this describes a 2d black hole. (IV)

For now, we will use Poincaré coordinates

In these coordinates: $\phi = \dfrac{\alpha + \gamma t + \delta(-t^2+z^2)}{z}$ ↝ 3 integration const!!

Here is where boundary conditions become important.

As we go to the boundary $z \to 0 \Rightarrow \phi \to \infty$
$h \to \infty$.

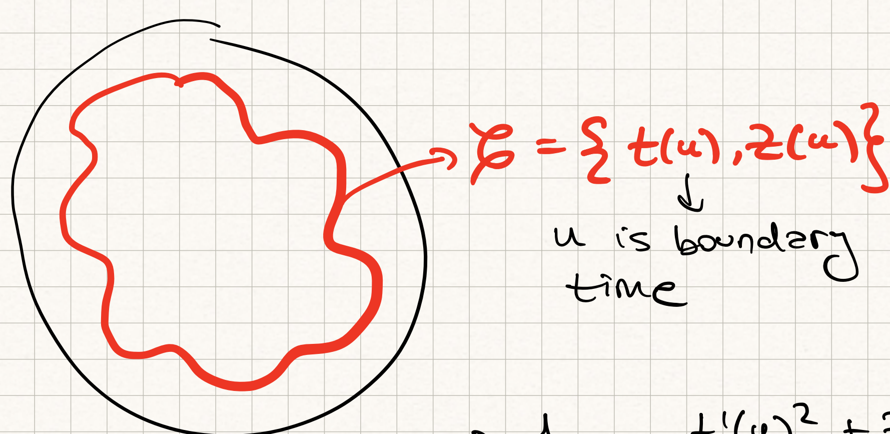
But we said we wanted $\phi_0 \gg \phi$ (note that ϕ_0 is anyways an external parameter)

⇒ the solution to this is imposing the boundary conditions and cutting off a piece of spacetime

(In the end there will be ε-independent qtities)

⊛ Implementing the boundary conditions

– How do we fix the length of the body?



For simplicity for now we will set $\hbar = 1$ (you can always rescale the cut-off)

$$\Rightarrow \frac{1}{\varepsilon^2} = \frac{t'(u)^2 + z'(u)^2}{z(u)^2}$$

$\Rightarrow$ Assuming $z>0$, this gives

$$\boxed{z(u)} = \varepsilon \sqrt{t'(u)^2 + z'(u)^2} = \boxed{\varepsilon\, t'(u)} + \mathcal{O}(\varepsilon^3)$$

$\Rightarrow$ So now all the theory depends on just one function $t(u)$.

Comment on symmetries:

* for any $t(u)$, the E-H part of the action would give the same result (we said it is topological)

  $\Rightarrow$ this can be interpreted as a full symmetry under reparametrisations of $u \to f(u)$

Most of this take you from one curve to another

$$g|_{bdy} = \frac{1}{\varepsilon^2} \rightsquigarrow \frac{du^2}{z^2} \rightsquigarrow \text{change } u \to \text{basically change the bdy cond} \to \text{change the curve}$$

However this is not the case for all curves: translations and rotations of a fixed shape around the hyperbolic disk do not change the part we are cutting.

$$t(u) \to \frac{a\,t(u) + b}{c\,t(u) + d}, \quad ad - bc = 1$$

form an $SL(2,\mathbb{R})$ subgroup of all reparametrisations and do not change the bdy.

⇒ All curves that cut the hyperbolic disk spontaneously break the reparametrisation symmetry down to $SL(2,\mathbb{R})$ (VI)

(Comment on $t(u)$ being a Goldstone mode)
↳ they all have the same action

2.2 The Schwarzian theory

* what is the action that rules the behaviour of $t(u)$??
  ↳ Here is where the JT action appeares w/ a term w/ a dilaton which will break the reparametrisation symmetry explicitly and give a finite action to $t(u)$.

* So what do we do? We integrate out ϕ

$$\Rightarrow S_{JT} = -\frac{1}{16\pi G_N}\left[\int d^2x \sqrt{g}\,(R+2)\phi + 2\int_{\partial M} \sqrt{h}\,\phi_b\, K\right]$$

(the first term) just says that $R=-2$ & we are left with

$$S_{JT} = -\frac{1}{8\pi G_N}\int_{\partial M} \phi_b K = -\frac{1}{8\pi G_N}\int_{\partial M} \frac{du}{\varepsilon}\frac{\tilde{\phi}}{\varepsilon} K$$

* What is K?? ~> curvature as seen from $\mathcal{M}$

VII

→ Usually K^a_{ij} (a → orthogonal, ij → surface). Usually the trace $K^a{}_{ii}$.

→ Matt Headrick: "Compendium of useful formulas"

Abstract: Almost everything I know

Remember we had $\mathcal{C} = \{t(u), z(u)\}$

This defines a tangent vector $T^a = (t', z')$

$$n^a = \frac{z}{\sqrt{t'^2 + z'^2}} (-z', t')$$

← note that n^a is normalised

$$\Rightarrow \quad K = -\frac{h_{ab} T^a T^c \nabla_c n^b}{h_{ab} T^a T^b} = \nabla_a n^a$$

(try using diffgeo.m package from Matt Headrick as well)

The result for any curve $\mathcal{C}$ in Poincaré coordinates is:

$$K = \frac{t'(t'^2 + z'^2 + z' z'') - z z' t''}{(t'^2 + z'^2)^{3/2}}$$

But we know that given our boundary conditions

$z(u) = \varepsilon\, t'(u) + \dots$, so we can evaluate

K as a function of $t'(u)$ only in the small ε limit.

The result is:

$$K = 1 + \varepsilon^2 \underbrace{\frac{2t't''' - 3t''^2}{2t'^2}} + \mathcal{O}(\varepsilon^4)$$

$\mathrm{Sch}(t(u),u) \to$ Schwarzian derivative

So now we can go back to the action

$$S_{JT} = -\frac{1}{8\pi G_N} \int \frac{du}{\varepsilon} \frac{\tilde{\Phi}(u)}{\varepsilon} \left(1 + \varepsilon^2 \,\mathrm{Sch}(t(u),u)\right)$$

For now let's assume that $\tilde{\Phi}(u) = \tilde{\Phi}$ constant

$$\Rightarrow \quad S_{JT} = -\frac{\tilde{\Phi}}{8\pi G_N} \int du \left(\frac{1}{\varepsilon^2} + \mathrm{Sch}(t(u),u)\right)$$

$\frac{1}{\varepsilon^2}$ ↓ divergent as $\varepsilon \to 0$ $\Rightarrow$ But not physical we can subtracted by local counter-terms

$$\Rightarrow S_{JT} = \dots \int \phi_b (K-1)$$

$$\Rightarrow \quad S_{JT} = \frac{-\tilde{\Phi}}{8\pi G_N} \int du \,\mathrm{Sch}(t(u),u)$$

this is the first important result of the day

SOME COMMENTS: * Gravity theory becomes a bdy thy.

$ad-bc=1$ $\quad$ * $\mathrm{Sch}(t(u),u) = \mathrm{Sch}\left(\frac{a\,t(u)+b}{c\,t(u)+d}, u\right)$

It is the lowest derivative local expression that is invariant under $SL(2,\mathbb{R})$ transformations.

* Now we can look for saddle-point solutions to this theory.

↳ the eom is: (HW)

$$\frac{\left[S(t(u),u)\right]'}{t'} = 0$$

$$= \left[\frac{1}{t'}\left(\frac{t''}{t'}\right)'\right]' = 0 \rightsquigarrow$$

Remember the three integration constants of the dilaton!!

⇓

bdy and bulk thys have the same information

So we need solutions with constant schwarzian.

(of course $SL(2,\mathbb{R})$ transformations will work but they are not physical, more like redundancies).

To find different solutions we use a property of the Sch derivative under composition

$$\Rightarrow S(f\circ g, t) = g'^2 S(f,g) + S(g,t)$$

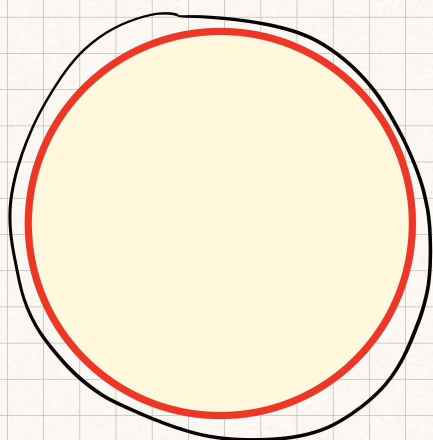
Now if $t(u) = \tan\frac{\tau(u)}{2}$ → note this takes us from Poincaré to Rindler coordinates.

$$\Rightarrow \boxed{S(t,u) = \frac{1}{2}\tau'(u)^2 + S(\tau(u),u)}$$

But now $\tau(u)$ goes from $[-\pi,\pi]$ so has periodicity 2π ⇒ they describe nice thermal solutions

It is easy to see that $\boxed{\tau(u) = \frac{2\pi}{\beta} u}$ is a (X) solution to $S(t,u) = \text{constant}$!! (It's not that easy to see that up to $SL(2,\mathbb{R})$ transformations it is the only solution).

So what's the solution that we found:



$\tau = \frac{2\pi}{\beta} u$ is a circle close to the geometrical body.

This is what we call a Euclidean black hole!!

How do black holes look in Euclidean signature??

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \quad , \quad f(r_h) = 0 .$$

Now we wick rotate time $\rightsquigarrow$ $ds_E^2 = f(r)\,dt_E^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$

Note that we still have a divergence at $r = r_h$ & $r = 0$.

But now let's take a look at the geometry close to the horizon:

$$\rho = 2\sqrt{\frac{r - r_h}{f'(r_h)}} \; ; \; \theta = \frac{t_E}{2} f'(r_h)$$

$$\Rightarrow \quad ds^2 \approx \underbrace{\rho^2 d\theta^2 + d\rho^2}_{} + r_h^2 \, d\Omega_2^2 + \dots$$

This is like the plane in polar coordinates if θ has the right periodicity.

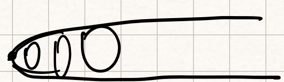
If θ does not have period 2π $\Rightarrow$ conical singularity

$$\Rightarrow \quad \theta \sim \theta + 2\pi$$

$$t_E \sim t_E + \beta = \frac{4\pi}{f'(r_h)} = \frac{1}{T}$$

$\Rightarrow$ so the geometry ends smoothly at $\underset{\sim}{\rho = 0}$

$r = r_h$



the cigar is the Euclidean BH $\rightarrow$ only sees up to the horizon $\rightarrow$ and has a thermal cycle that is contractible

$\Rightarrow$ the hyperbolic disk is the AdS_2 version of this and the thermal cycle reduces smoothly to zero at the center of the disk $\Rightarrow$ the solution we found <u>is</u> a Euclidean BH solution.

2.3 Thermodynamics

* On-shell action gives the partition function of the bdy theory at finite temperature

$$Z(\beta) = e^{-S_{grav}}$$

But this is very simple to evaluate at the saddle point:

$$S_{grav} = S_{sch} = -\frac{1}{8\pi G_N} \int_0^\beta du \, \tilde{\Phi}(u) \left(S(\tau,u) + \frac{1}{2} \tau'(u)^2 \right)$$

But $\tau(u) = \frac{2\pi}{\beta} u$

$$\Rightarrow S_{sch} = -\frac{\tilde{\Phi}}{8\pi G_N} \int_0^\beta du \left(\frac{2\pi}{\beta}\right)^2 \cdot 1 = -\underbrace{\frac{4\pi^2 \tilde{\phi}}{8\pi G_N}}_{c} \frac{1}{\beta}$$

$\Rightarrow$ Including the topological term:

$$S_{grav} = -S_0 - c/\beta .$$

$\Rightarrow$ Now we identify $\log Z \equiv -\beta F$

$$\Rightarrow S_{th} = (1 - \beta \partial_\beta) \log Z = S_0 + c/\beta$$

$$c(\beta) \equiv \beta^2 \partial_\beta^2 \log Z = c/\beta$$

$\Rightarrow$ We recover the linear in T behaviour. Note this is only possible due to the appearance of Φ : dimensionful parameter!]

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(2.4) An action for the fluctuations

When can ask what happens if we perturb a little away from the saddle (assume $\beta = 2\pi$)

$$\Rightarrow \quad \tau(u) = u + \varepsilon(u) \quad w/ \quad \varepsilon \ll 1$$

$$\Rightarrow S(\tau, u) + \frac{\tau'^2}{2} = \frac{1}{2} + \varepsilon' + \varepsilon'' + \left(\frac{1}{2}\varepsilon'^2 - \frac{1}{2}\varepsilon''^2 - (\varepsilon''\varepsilon')'\right) + \mathcal{O}(\varepsilon^3)$$

$(\varepsilon''\varepsilon')'$ → TOT. DER.

$$\Rightarrow \boxed{I_{Sch} = \frac{c}{2}\int_0^{2\pi} du \left(\varepsilon''^2 - \varepsilon'^2\right)}$$

Now we go to Fourier space $\varepsilon(u) = \sum_{n\in\mathbb{Z}} \varepsilon_n e^{inu}$

$$\Rightarrow \quad I_{Sch} = \frac{c}{2}\sum_{n\in\mathbb{Z}} (n^4 - n^2)\, \varepsilon_n \varepsilon_{-n}$$

There are 3 zero modes $n = 0, \pm 1 \Rightarrow$ non-invertible

But they are the 3 $SL(2,\mathbb{R}) \Rightarrow$ not physical so we don't integrate over them.

$\Rightarrow$ the propagator we get is:

$$\langle \varepsilon(u)\varepsilon(0)\rangle = \sum_{n\neq 0,\pm 1} \frac{e^{inu}}{n^2(n^2-1)}$$

→ in 2 weeks we will use it to compute the 4pt function

2.5 SUMMARY

(a) we studied JT gravity

(b) It becomes a bdy theory of reparametrisations w/ a Schwarzian action.

(c) we found saddle points of this action at finite temperature.

(d) the thermodynamics for those saddles gives a linear-in-T specific heat as expected from near-extremal bh's.

NEXT: Will see how this same pattern appears in a theory of quantum mechanics called SYK.

HOMEWORK PROBLEMS:

① Find the equation $T^{\phi}_{\mu\nu} = 0$ in Poincaré coordinates and verify that $\phi = \dfrac{\alpha + \gamma t + \delta(t^2 + z^2)}{z}$ is the most general solution.

② For a curve $\mathcal{C} = (t(u), z(u))$ in Poincaré coordinates, show that

$$K = \frac{t'(t'^2 + z'^2 + z' z'') - z z' t''}{(t'^2 + z'^2)^{3/2}}$$

(maybe Mathematica can help w/ that)

(3) Verify that for a curve "close" to the boundary of AdS_2, this reduces to $K = 1 + \varepsilon^2 \, Sch(t(u), u)$, where $\varepsilon \ll 1$ is the same small parameter as in the notes.

(4) Convince yourself that if $ad - bc = 1$, then

$$Sch(t(u), u) = Sch\left(\frac{a\,t(u) + b}{c\,t(u) + d}, u\right)$$

(5) Expand the schwarzian around the solution $t(u) = u$ $(t(u) = u + \varepsilon(u))$ and find that to quadratic order and up to total derivatives, it gives:

$$Sch(u + \varepsilon(u), u) = (\varepsilon''^2 - \varepsilon'^2) + O(\varepsilon^3)$$