LonTI lectures: on SYK and the emergence of spacetime

Lecture 3: The SYK model

For this lecture we will "forget" about what we know about gravity and study a problem in quantum mechanics. AIM: See the emergence of spacetime

TODAY:
* Define & study the SYK model
* Compute two-point functions in the large-N limit.
* See the emergence of the Schwarzian theory in the IR.

3.1 The Sachdev-Ye-Kitaev model

* Simple, finite, quantum mechanical theory involving N Majorana fermions.

3.1.1 Majorana fermions in QM

$$\boxed{\psi_i \,,\quad i=1,\dots,N. \quad \text{Majorana} \quad \psi_i^\dagger=\psi_i \,. \quad \{\psi_i,\psi_j\}=\delta_{ij} \quad i,j=1,\dots,N}$$

We will work in Euclidean signature again and so we need to find representations of this Clifford algebra.

there's a simple way of finding them:

Let's define:

(we assume $N = 2k$ is even)

$$\left.\begin{aligned} c_i &= \frac{1}{\sqrt{2}}\left(\psi_{2i-1} - i\,\psi_{2i}\right) \\ c_i^{+} &= \frac{1}{\sqrt{2}}\left(\psi_{2i-1} + i\,\psi_{2i}\right) \end{aligned}\right\} \; i = 1, \dots, k$$

$\Rightarrow$ these ones satisfy the "usual" fermionic anti-commutation relations: $\{c_i, c_j\} = \{c_i^{+}, c_j^{+}\} = 0$; $\{c_i, c_j^{+}\} = \delta_{ij}$.

So now we know how to build a rep of this. We choose a vacuum annihilated by all the $c_i|0\rangle = 0$

$\Rightarrow$ a basis for this representation is given by $(c_1^{+})^{n_1}(c_2^{+})^{n_2}\dots(c_k^{+})^{n_k}|0\rangle$ $\quad n_k = 0, 1$

$\Rightarrow$ How many states do we have? each n_i can be 0 or 1 and we have k of them $\Rightarrow 2^k = 2^{N/2}$

(It is possible to prove that this is the only irep up to unitary equivalence)

NOTE: Hilbert space grows exponentially with N

$\Rightarrow$ What are these fermions? We can build them recursively

- Start with $N = 2$ $\rightsquigarrow$ $\psi_1^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\psi_2^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(Pauli matrices
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)

- For any k $\Rightarrow$ $\psi_i^{(k)} = \psi_i^{(k-1)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$i = 1, \dots, N-2$

$$\psi_{N-1}^{(k)} = \frac{1}{\sqrt{2}}\, I_{2^{k-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\psi_N^{(k)} = \frac{1}{\sqrt{2}}\, I_{2^{k-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(*) Basically, $\boxed{\text{each fermion is a } 2^{N/2} \times 2^{N/2} \text{ matrix}}$

↳ mention Kolya's lecture

For instance, $N=4$

$$\begin{pmatrix} 0 & 0 & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & -i/\sqrt{2} \\ i\sqrt{2} & 0 & 0 & 0 \\ 0 & -i/\sqrt{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ +\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

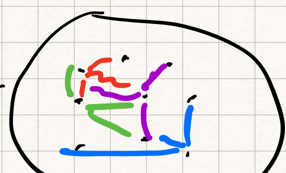
HW: show that $N=2$ & 4 satisfy the Majorana condition.

HW (computationally): Create the $N=20$ representation. What is the largest N you can get??

The SYK Hamiltonian:

$$\boxed{H_{SYK} = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \psi_i \psi_j \psi_k \psi_l}$$

diff colours
diff couplings



But we do all possible interactions between groups of 4 → these are called "all-to-all" couplings.

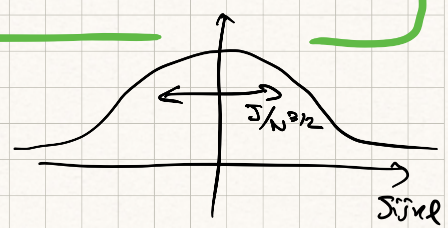
HW: Show that the number of independent couplings needed is $\frac{N!}{4!\,(N-4)!}$ and in the large-N, it grows as $\sim \frac{N^4}{24}$.

$\Rightarrow$ So, H_{SYK} is also a $2^{N/2} \times 2^{N/2}$ matrix.

But I am missing a key ingredient: What are the J_{ijkl}??

* We will choose the couplings randomly from a Gaussian ensemble with mean $\bar{\mu} = 0$ and variance

$$\sigma = \frac{\sqrt{3!}\, J}{N^{3/2}}$$



* J is fixed and has dimensions of energy.
* $\sqrt{3!}$ is normalisation but the scaling with N will be important to have a proper large-N limit.
* Of course, couplings will be "arbitrary", so instead we will average over all possible realizations of the model. (even though the model is self-averiging so in many cases it is not necessary)

At an operational level:

1st. choose random couplings

(2): compute whatever you are interested in computing

(3) Chose another set of couplings

(4) Repeat ...

(5) Compute an average over all results.

Caution: say you want to compute $\log Z$

What do you do? $\langle \log Z \rangle_J$ or $\log \langle Z \rangle_J$??

more complicated involves usually a replica trick

"annealed" disorder

But: For SYK at large-N both approaches give the same result, so we will do annealed disorder for the rest of the lecture.

Finally, the number of fermions in the interaction is arbitrary, so sometimes it is convenient to talk about the q-SYK model:

$$\boxed{H_q = i^{q/2} \sum J_{i_1 \dots i_q} \psi_{i_1} \dots \psi_{i_q}}$$

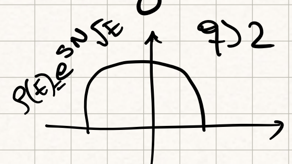
$\sigma \simeq J/N^{\frac{q-1}{2}}$

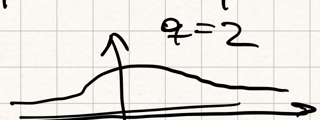
* Analitically solvable when $q \to \infty$
* $q=2$ is fermions with random masses (not chaotic)

3.1.1 The Energy Spectrum of H_{SYK}

HW: Diagonalise the H_{SYK} and plot the eigenvalues in a histogram. Compare all q's w/ $q=4$.

Plots & reference to Kolya





Semi-circle of random theories $\to$ many states close to the ground state

long tail for $q=2$ $\to$ integrable model

Last comment: we can compute now 2pt-functions

e.g.: $\langle \psi_i(\tau)\psi_i(0)\rangle_\beta = \mathrm{Tr}\left[e^{-\beta H} e^{\tau H}\, \psi_i\, e^{-\tau H}\, \psi_i\right]$

$\Rightarrow$ It's just matrix multiplication + Trace

[see Vadya's lecture]

3.2 Solving the SYK model in the large-N limit

- Two ways: (1) Diagrammatic expansion
  (2) Path integral

Now we forget about matrices and we treat SYK as a 0+1d QFT.

So it is better to work in the action formalism:

$$\boxed{S_{SYK} = \int d\tau \left[\frac{1}{2}\sum_i \psi_i \partial_\tau \psi_i + \sum J_{ijkl}\, \psi_i\psi_j\psi_k\psi_l\right]}$$

$\Rightarrow$ We also need to add disorder

$$\langle Z\rangle_J \doteq \int dJ_{ijkl}\, e^{-\sum \frac{J_{ijkl}^2}{N^3}} \int D\psi_i\, e^{-S_{SYK}}$$

Let's do first perturbation theory in J.

* $\boxed{J=0} \Rightarrow H=0 \Rightarrow \psi_i(t) = \psi_i$ but given we have

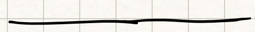
$\{\psi_i, \psi_j\} = \delta_{ij} \Rightarrow G_{ij}^{free}(\tau) = \frac{1}{2}\delta_{ij}\,\mathrm{Sign}(\tau)$

$$\Rightarrow \boxed{G^{free}(\tau) = \frac{1}{N}\sum G_{ii}^{free} = \frac{1}{2}\mathrm{Sgn}(\tau)}$$

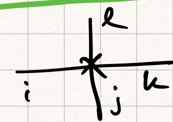
$\rightarrow$ Time-ordered 2pt-function. Anti-symm because fermions

$$G^{free}_{ij}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, G^{free}_{ij}(\tau) = -\frac{\delta_{ij}}{i\omega}$$

* $\boxed{\text{Propagator and vertices}}$ For each realisation of the model we have



free propagator



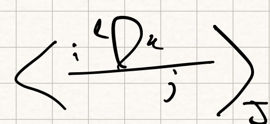
a 4pt vertex

* But we also need to average over $\boxed{\text{disorder}}$ $\Rightarrow$ we will consider J_{ijkl} as a "field" but with no free propagation zero 1pt function and:

$\langle J_{i_1 j_1 k_1 l_1} J_{i_2 j_2 k_2 l_2} \rangle_J = \frac{3!\, J^2}{N^3} \delta_{i_1 i_2} \delta_{j_1 j_2} \delta_{k_1 k_2} \delta_{l_1 l_2}.$

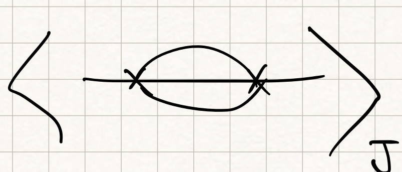
* Now we can start doing diagrams:

- First contribution vanishes



$\sim \langle J_{ijkl} \rangle = 0$

- Next diagram is the melon



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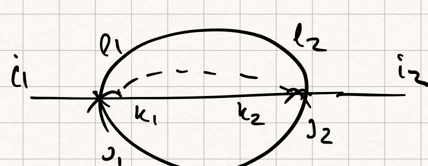


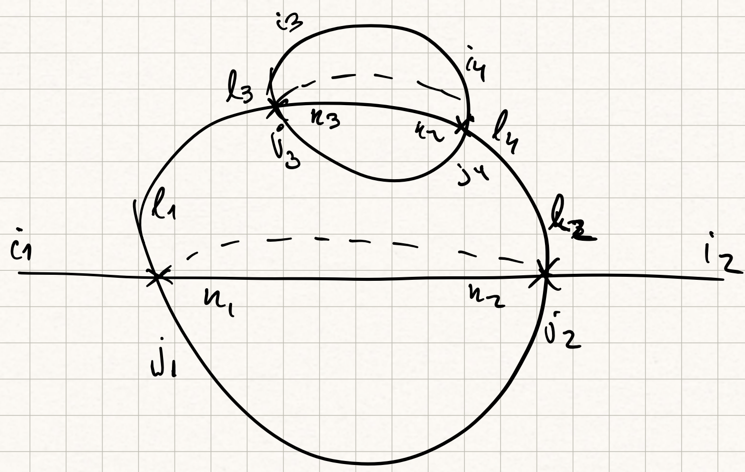
$$= 3!\,\frac{J^2}{N^3}\, \delta_{i_1 i_2} \delta_{j_1} \delta_{j_2} \delta_{k_1 k_2} \delta_{l_1 l_2}\, G_{l_1 l_2} G_{k_1 k_2} G_{j_1 j_2}$$

$$= 3!\,\frac{J^2}{N^3}\, G^{free}_{jj}\, G^{free}_{kk}\, G^{free}_{ll}\, \delta_{i_1 i_2}$$

$$= 3!\, J^2\, \frac{\cancel{N^3}}{\cancel{N^3}} \left(G^{free}\right)^3 \delta_{i_1 i_2}$$

Does not scale with N!!

- J^3 will vanish again, so we can go to J^4



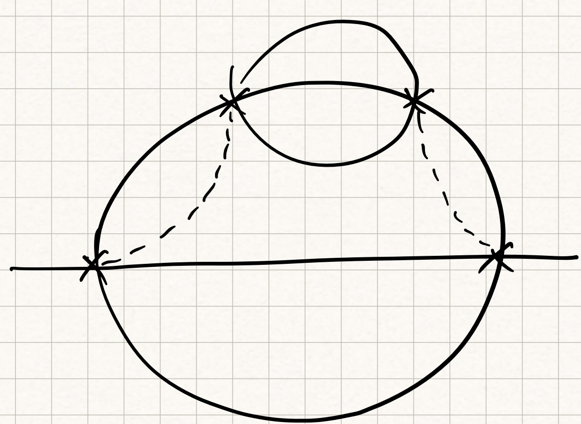
$$\frac{J^4}{N^6}\ \delta_{i_1 i_2}\,\delta_{j_1 j_2}\,\delta_{k_1 k_2}\,\delta_{l_1 l_2}\,\delta_{l_3 l_4}\,\delta_{j_3 j_4}\,\delta_{k_3 k_4}\,\delta_{i_3 i_4}$$
$$G_{l_1 l_3}\,G_{k_1 k_2}\,G_{j_1 j_2}\,G_{l_2 l_4}\,G_{k_3 k_4}\,G_{j_3 j_4}\,G_{i_3 i_4}$$

$$\Rightarrow \frac{J^4}{N^6}\,\delta_{i_1 i_2}\left(G^{free}_{jj}\right)^5 \underbrace{\delta_{l_1 l_2}\,\delta_{l_3 l_4}\,G_{l_1 l_3}\,G_{l_2 l_4}}$$

$$\left.\begin{matrix} G_{l_1 l_3}\,G_{l_1 l_3} \\ \downarrow \\ \propto \delta_{l_1 l_3} \end{matrix}\right\} G_{ll}$$

$$\Rightarrow \frac{J^4}{N^6}\,\delta_{i_1 i_2}\left(G^{free}_{ii}\right)^6 = J^4 \cdot \underline{\underline{N^0}}$$

HW



Show that this diagram is subleading in the large-N expansion

$$= \frac{J^4}{N^6}\left(G^{free}_{kk}\right)^4 G_{k_1 k_2}$$

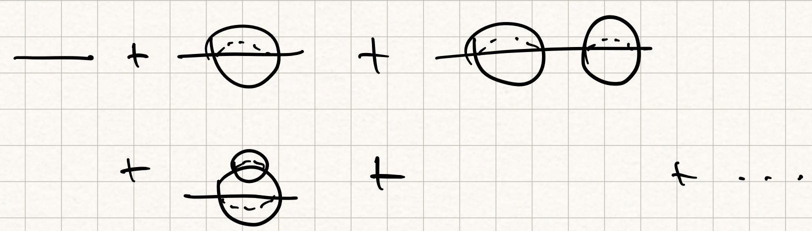
(similar to planar limit in SU(N)) $= \dfrac{J^4}{N^2}$

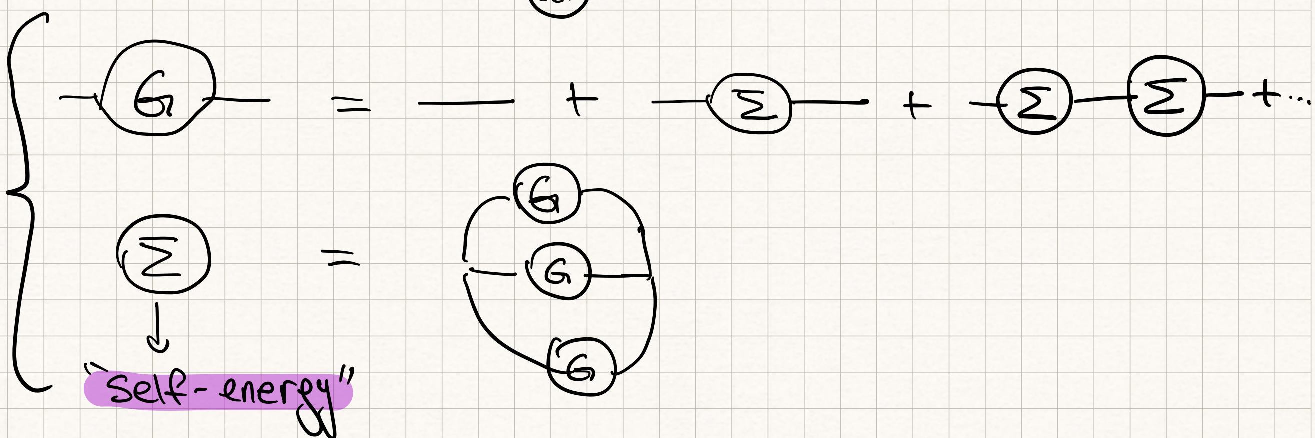
CONCLUSION: In the large-N limit only melonic diagrams contribute!!

(Note the importance of the N-scaling in the Variance)

Now let's write down the two-point function expansion:

$\Rightarrow$ $G(\tau)$ =





We would like to write these diagrams as equations

But we need to be careful $G(\tau_1, \tau_2)$ are bilinear functions $\Sigma(\tau_1, \tau_2)$

$\Rightarrow$ multiplication is like matrix multiplication

$$A \cdot B(\tau, \tau') = \int d\tau'' \, A(\tau, \tau'') B(\tau'', \tau')$$

$$(A \cdot B)_{ij} = \sum_k A_{ik} B_{kj}$$

$\Rightarrow$ the 1st line reads

$$G = G^{free} + G^{free}\,\Sigma\, G^{free} + G^{f}\,\Sigma\, G^{f}\,\Sigma\, G^{f} + \dots$$

$$= G^{free}\left[1 + \Sigma G^{f} + \Sigma G^{f}\,\Sigma G^{f} + \dots\right]$$

$$= G^{f}\left[1 + \Sigma G^{f}\right]^{-1}$$

$\hookrightarrow \sum (\Sigma G)^{k} = \dfrac{1}{1-\Sigma G}$

$$G = \left[(G^{free})^{-1} - \Sigma\right]^{-1}$$

But we know $(G^{free})^{-1} = \delta(\tau,\tau')\partial_{\tau'}$

$\Rightarrow$ We usually write this equation as

$$\boxed{\begin{aligned} G &= (\partial_\tau - \Sigma)^{-1} \\ \Sigma &= J^2 G^3 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} G &= (\partial_\tau - \Sigma)^{-1} \\ \Sigma &= J^2 G^{q-1} \end{aligned}}$$

$\Rightarrow$ These eqs. are called Schwinger-Dyson (SD) equations.

$\Rightarrow$ We managed to "solve" the theory at large-N at all orders in J !!

$\hookrightarrow$ (Of course we still need to solve these eqs. but it is already important that we found such eqs. to start with. Usually, it is not the case in QFT.)