* The Path integral approach

Alternatively, we can find the SD eqs from a saddle-point limit of the full path integral. the calculation is a bit long but it is quite clear.

$$\langle Z\rangle_J = \int D\psi_i \, dJ_{ijkl} \, e^{-\sum J_{ijkl}^2/J^2_{N^3}} \, e^{-\left(\int d\tau \frac{1}{2}\psi_i\partial_\tau\psi_i + \sum J_{ijkl}\psi_i\psi_j\psi_k\psi_l\right)}$$

- Note first that J_{ijkl} only appear quadratically (and it is a normal integral so we can do it!)

$$\int dx \, e^{-ax^2+bx} = \sqrt{\pi/a}\; e^{b^2/4a}$$

$$\Rightarrow \langle Z\rangle_J \sim \int D\psi_i \, \exp\left(-\int d\tau \frac{1}{2}\psi_i\partial_\tau\psi_i + \sum_{1\le i<j<k<l\le N} \frac{3J^2}{N^3}\iint d\tau d\tau' \, \psi_i\psi_j\psi_k\psi_l(\tau)\,\psi_i\psi_j\psi_k\psi_l(\tau')\right)$$

Now we can use the fermions algebra to show that

$$\sum_{1\le j<j<k<l\le N} \psi_i\psi_j\psi_k\psi_l(\tau)\,\psi_i\psi_j\psi_k\psi_l(\tau') = \frac{1}{4!}\left[\sum_i \psi_i(\tau)\psi_i(\tau')\right]^4$$

* Here comes the clever trick. We will insert an identity in the path integral to make sure that $G = \psi_i\psi_i$

$$1 = \int DG \; \delta\left(N G(\tau,\tau') - \sum \psi_i(\tau)\psi_i(\tau')\right)$$

$$\sim \int DG\, D\Sigma \, \exp\left(-\frac{N}{2}\iint d\tau d\tau' \, \Sigma\left(G - \frac{1}{N}\sum\psi\psi\right)\right)$$

$\Rightarrow$ G is the fermion bilinear & Σ is a Lagrange multiplier to impose the constraint!!

* Now we put everything together

$$\langle Z \rangle_J \sim \int D\psi_i \, DG \, D\Sigma \, \exp\Big(-\frac{1}{2} \sum_i \psi_i \partial_\tau \psi_i - \frac{1}{2} \iint N \, \Sigma \Big(G - \frac{1}{N} \sum_i \psi_i \psi_i \Big) + \frac{J^2 N}{2 \cdot 4} \iint G^4 \Big)$$

$\Rightarrow$ Now the fermions appear quadratically inside an exponential, so we can integrate them out!!

Wikipedia $\leftarrow$ Berezin Gaussian Integral

$$\int D\psi \, e^{-\frac{1}{2} \psi A \psi} = \sqrt{\det A} \, .$$

$$\boxed{\begin{aligned} \langle Z \rangle_J &\sim \int DG \, D\Sigma \, \det(\partial_\tau - \Sigma)^{N/2} \exp\Big\{ -\frac{N}{2} \iint \Big(\Sigma G - \frac{J^2}{4} G^4 \Big) \Big\} \\ &= \int DG \, D\Sigma \, e^{-N \, I[G, \Sigma]} \\ \text{with } \; I(G, \Sigma) &= -\frac{1}{2} \log \det(\partial_\tau - \Sigma) + \frac{1}{2} \iint \Big(\Sigma G - \frac{1}{4} J^2 G^4 \Big) \end{aligned}}$$

For general q

$$I(G, \Sigma) = -\frac{1}{2} \log \det(\partial_\tau - \Sigma) + \frac{1}{2} \iint \Big(\Sigma G - \frac{1}{q} J^2 G^q \Big)$$

Now, because we have an N in front we can take $N \to \infty$ and do a saddle-point approximation.

HW Show that the saddle-point eqs are the same as the SD-eqs. obtained before.

3.3 Solutions to SD-eqs

$$\begin{cases} G = (\partial_\tau - \Sigma)^{-1} \\ \Sigma = J^2 G^{q-1} \end{cases} \Leftarrow$$

Nikolay will solve them numerically

How this depends on the coupling?

Free fermions

$$\text{UV} \longrightarrow \text{IR} \quad ??$$

UV $(J=0)$ IR $J(\tau-\tau')\gg 1$

$\downarrow$

$\Sigma=0 \Rightarrow G=\partial_\tau^{-1}$

$\Rightarrow \partial_\tau$ is a UV term.

$\hookrightarrow \delta(\tau-\tau')\partial_\tau$

So let's see what happens if we remove that UV-term.

$$\boxed{\text{CRUCIAL OBSERVATION: the equations acquire an extra symmetry}}$$

Suppose $\tau \to \phi(\tau)$ s.t.

$$\boxed{\begin{aligned} G(\tau,\tau') &\to [\phi'(\tau)\phi'(\tau')]^{\Delta} G(\phi(\tau),\phi(\tau')) \\ \Sigma(\tau,\tau') &\to [\phi'(\tau)\phi'(\tau')]^{\Delta(q-1)} \Sigma(\phi(\tau),\phi(\tau')) \end{aligned}}$$

$\Rightarrow$ Under reps, both Σ & G transform as conformal 2 pt. functions.

HW Show that $\Delta = \frac{1}{q}$.

Show that the first SD eq. is invariant.

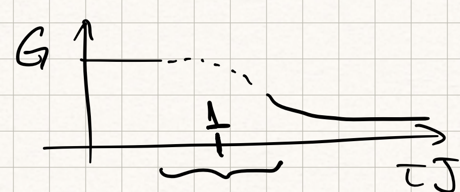
$\Rightarrow$ EMERGENT CONFORMAL SYMMETRY IN THE IR THAT WILL BE EXPLICITLY BROKEN BY THE ∂_τ TERM!!

* So now we can look at explicit solutions (on the line):

$$\Rightarrow G_c(\tau) = \# \frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}}, \quad \Sigma_c = \#^{q-1} J^2 \frac{\mathrm{sgn}(\tau)}{|\tau|^{\Delta(q-1)}}$$

$\Rightarrow$ They solve the SD eqs. in the $\underline{\underline{\text{IR}}}$. ($\tau \gg J^{-1}$)

(in the UV, $\tau \ll J^{-1} \Rightarrow G = \mathrm{sgn}(\tau)$)



here we don't have analytic expression.

* At finite temp $\tau \sim \tau + \beta$ $\quad \phi(\tau) = \tan \frac{\pi \tau}{\beta}$

$$\Rightarrow G_c = \# \left[\frac{\pi}{\beta \sin \frac{\pi \tau}{\beta}} \right]^{2\Delta} \mathrm{sgn}(\tau).$$

* Symmetries: Now that we found a solution, not all the reparametrisations $\phi(\tau)$ leave the solution invariant. Only $SL(2,\mathbb{R})$ transformations $\phi(\tau) = \frac{a\tau + b}{c\tau + d}$, $ad - bc = 1$

$\Rightarrow$ the solution breaks the full reparametrisation symmetry down to $SL(2,\mathbb{R})$.

3.4 The emergence of the Schwarzian

(Rosenhaus 1807.08334)

Up to now, we found an entire space of solutions

$$G(\tau,\tau') = b\,\frac{\text{sign}(\tau_{12})}{J^{2\Delta}}\;\frac{\phi'(\tau)^{\Delta}\,\phi'(\tau')^{\Delta}}{|\phi(\tau)-\phi(\tau')|^{2\Delta}}.$$

But now we would like to move a bit from the conformal point $\Rightarrow$ we need to include the effect of the ∂_τ term. Let's go back to the action:

$$I(\Sigma,G) = -\frac{1}{2}\log\det(\partial_\tau-\Sigma)+\frac{1}{2}\int d\tau_1 d\tau_2\left(\Sigma G-\frac{J^2}{4}G^4\right)$$

Just for convenience let's define $\sigma(\tau,\tau') = \delta(\tau-\tau')\partial_\tau$ and let's do the change of variables to $\Sigma\to\Sigma+\sigma$.

$$\Rightarrow I(\Sigma,G) \to I_{CFT} + I_S \quad \text{where}$$

$$I_{CFT} = -\frac{1}{2}\log\det(-\Sigma)+\frac{1}{2}\int d\tau_1 d\tau_2\left(\Sigma G-\frac{J^2}{4}G^4\right)$$

$$I_S = \frac{1}{2}\int d\tau_1 d\tau_2\, \sigma(\tau_1,\tau_2)\, G(\tau_1,\tau_2)$$

We want to see the leading correction that I_S gives when we move away from the ∞ IR.

Since σ has a delta function, it is convenient to change times to $\frac{\tau_1 - \tau_2}{2} = \tau_{12}$

$$\frac{\tau_1 + \tau_2}{2} = \tau_+$$

$\Rightarrow$ then we can expand $G(\tau_1, \tau_2)$ for $\tau_{12} \ll 1$

the result is $G \to \frac{\mathrm{sgn}(\tau_{12})}{|J\tau_{12}|^{2\Delta}} \left(1 + \frac{\Delta}{6} \tau_{12}^2 \, \mathrm{Sch}(\phi(\tau_+), \tau_+) \right)$

$$\Rightarrow \boxed{I_S \to \frac{\#}{J} \int d\tau_+ \, \mathrm{Sch}(\phi(\tau_+), \tau_+) + \ldots}$$

NOTE: the number cannot be determined by this procedure $\sim \underbrace{\int d\tau_- \, \delta(\tau_-) \partial_\tau \tau_-^{2-2\Delta}}_{\text{needs to be regulated very UV term.}}$

SUMMARY: SYK model of fermions in QM w/disorder

- $\hookrightarrow$ exact diagonalisation
- $\hookrightarrow$ S-D large-N
- $\hookrightarrow$ Analytic in the IR

} Very solvable model

- In the large-N limit, there's an emergent reparametrisation symmetry in the IR that is broken spontaneously by the conformal solution

& explicitly by the Schwarzian action.

- Very similar to the $nAdS_2$ story.

NEXT:

- Solve SYK completely & analytically in the large-q limit.
- See the linear-in-T specific heat
- Quantum chaos & out-of-time ordered 4 pt. functions.

HOMEWORK EXERCISES:

1. Representations of Clifford algebra

   a. show that $N=2$ & 4 in the notes satisfy the Majorana condition.

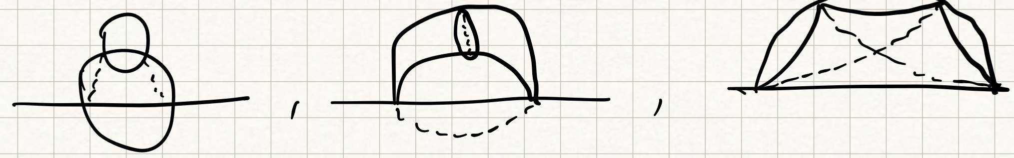
   b. (computationally): create the $N=20$ representation. what is the largest N you can get??

2. Show that the number of independent couplings needed is $\frac{N!}{4!(N-4)!}$ and in the large-N, it grows as $\sim \frac{N^4}{24}$. what is the generalisation to general q?

3. With Kolya's help:

   Diagonalise the H_{SYK} and plot the eigenvalues in a histogram. compare all q's w/ $q=2$.

(4) Show that the following diagrams are subleading in the large-N expansion:



If you want even more diagrams look at figure 3 in 2002.12187.

(5) a) Show that the saddle-point eqs. of the G, Σ actions are
$$\begin{cases} G = (\partial_\tau - \Sigma)^{-1} \\ \Sigma = J^2 G^{q-1} \end{cases}$$

b) (extra: useful for next lecture)

If
$$\begin{cases} G = g_0 + \frac{g}{q} + \dots \\ \Sigma = \Sigma_0 + \frac{\Sigma}{q} + \dots \end{cases}$$
, show that SD eqs. reduce to a single differential equation
$$\partial_\tau^2 g \simeq J^2 e^{g(\tau)}$$

HINT: It might be useful to go to Fourier space.

(6) By putting the conformal ansatz into the S-D eqs. show that $\Delta = 1/q$ & $b^q = \frac{1}{\pi J^2}\left(\frac{1}{2} - \frac{1}{q}\right)\tan\frac{\pi}{q}$

for $G_c = \frac{b}{|\tau|^{2\Delta}} \operatorname{sgn}(\tau)$.

(7) By now, you should be able to compute a number of different two point functions:

* Exact diagonalization at finite-N

* Solution to SD at large-N
* Conformal two-point function.
* large-q 2-pt function if you did 5b)

Plot the 4 of them for different values of βJ & q.

Do they start looking similar when $\beta J \gg 1$? Why or why not??