London Theory Institute Lectures Series

Nikolay Gromov

“ From spin chain to AdS/CFT with Mathematica ”

Informations

Pre-recorded Lectures : Youtube

Introduction to Mathematica : Youtube

Live Tutorial : Monday 8th of March, 10h30

Abstract

In this introductory lecture we describe the XXX Heisenberg spin chain, study its spectrum, wavefunctions and discuss integrability of the system. Some examples are given with simple Mathematica code. We also discuss applications to AdS/CFT correspondence.

Some derivations are formulated in the form of step-by-step exercises. They can be solved either with Mathematica or by hand. A quick introduction to Mathematica is provided in a separate video.

The problem has 3 parts, which are totally independent, you can solve in any order.

Part 1 Direct diagonalization and Bethe Ansatz equations (Mathematica)

Use the code from the lecture to

1) Construct explicitly L=6 Hamiltonian, and find its eigenvalues. You should obtain:

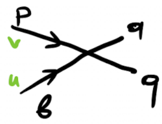
E	degeneracy
0	7
1	10
2	3
3	10
4	7
5	6
6	1
$5-\sqrt{5}$	3
$5+\sqrt{5}$	3
$5-\sqrt{13}$	1
$5+\sqrt{13}$	1
$\frac{1}{2}(7-\sqrt{17})$	6
$\frac{1}{2}(7+\sqrt{17})$	6

2) Solve BAE for 1 magnon, 2 magnon. You should reproduce most of the spectrum correctly (but not the degeneracies)
3) Try to solve 3 magnon case

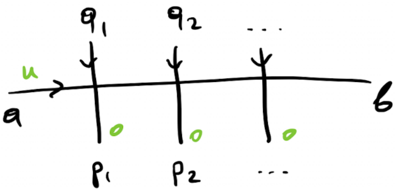
Part 2 Algebraic Bethe ansatz (by hand)

Using the following conventions:

$$R^{aq}_{bp}(u-v) = \left(u-v-\frac{i}{2}\right)\delta^{a}_{b}\delta^{q}_{p} + i\,\delta^{a}_{p}\delta^{q}_{b} =$$



$$T_{ab}(u) =$$



$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{ab}$$

1) Use TTR=RTT relation to derive:

$$A(v)\,B(u) = f(u-v)\,B(u)\,A(v) + g(v-u)\,B(v)\,A(u)$$
$$D(v)\,B(u) = f(v-u)\,B(u)\,A(v) + g(u-v)\,B(v)\,D(u)$$

Where $f(u)=1+\frac{i}{u}$ $\quad g(u)=\frac{i}{u}$

2) Prove that action on the vacuum (all spins up) is given by:

$$A(u)\,|\Omega\rangle = \left(u+\frac{i}{2}\right)^L |\Omega\rangle \qquad D(u)\,|\Omega\rangle = 0$$
$$D(u)\,|\Omega\rangle = \left(u-\frac{i}{2}\right)^L |\Omega\rangle$$

3) Prove that one magnon state can be written as:

$$\Psi = B(u_1)\,|\Omega\rangle$$

4) Try to prove the general case, of any number of magnons the eigenstate of A+D is:

$$\Psi = B(u_1)\,\ldots\ldots\,B(u_M)\,|\Omega\rangle$$

When the roots satisfy the Bethe ansatz equation.

Part 3 AdS/CFT applications (Mathematica)

Circular string vs Bethe ansatz (Simple)

Exercises for the lecture of Nikolay Gromov

Theory

In this exercise we check the AdS/CFT duality. Namely we compare energy of classical string moving in $AdS^5 \times S_5$ with a dual quantity in Super-Yang-Mills theory. We will check numerical that two very different quantities match.

Strong coupling (Classical string theory)

- Check (without Mathematica) that the equations of motion are

$$\partial^a \partial_a \vec{X} = -\vec{X}(\partial_a \vec{X} \partial^a \vec{X})$$

The Polyakov action for the string moving inside S^3 is given by

$$L = \int d\tau \int_0^{2\pi} d\sigma [\partial_a \vec{X}_i \partial^a \vec{X} - \Lambda(X^2 - 1)]$$

where $\vec{X}$ is a four component vector and Λ is the Lagrange multiplier to ensure that $\vec{X} \in S^3$.

The energy of the string is

$$\Delta = \sqrt{\lambda}\sqrt{(\partial_\sigma \vec{X})^2 + (\partial_\tau \vec{X})^2}$$

where λ is the ‘t Hooft coupling

- Check that

$$X_1 = \sqrt{\frac{\mathcal{J}(1-\alpha)}{w_1}} \cos(w_1 \tau + \alpha n \sigma) \ , \ X_2 = \sqrt{\frac{\mathcal{J}(1-\alpha)}{w_1}} \sin(w_1 \tau + \alpha n \sigma)$$

$$X_3 = \sqrt{\frac{\mathcal{J}\alpha}{w_2}} \cos(w_2 \tau + (\alpha - 1) n \sigma) \ , \ X_4 = \sqrt{\frac{\mathcal{J}\alpha}{w_2}} \sin(w_2 \tau + (\alpha - 1) n \sigma)$$

is a solution of the equations of motion. Derive the relation between w_1, w_2 and $\mathcal{J}, \alpha$, n

- One can try to find w_1, w_2 as functions of the other parameters which is very complicated. Instead find w_1, w_2 perturbatively for large $\mathcal{J}$ i.e. find a and b in the expansion

$$w_1 = \mathcal{J} + \frac{a}{\mathcal{J}} + \dots \ , \ w_2 = \mathcal{J} + \frac{b}{\mathcal{J}} + \dots$$

such that the equations of motion are satisfied up to a terms $\sim 1/\mathcal{J}$
Hint: you should find $a = \alpha(2\alpha - 1)n^2$ up to a simple multiplier

- Compute the energy Δ also as an expansion in $\mathcal{J}$. Denote $L = \sqrt{\lambda}\mathcal{J}$. The result you should get is

$$L - \frac{(\alpha - 1)\alpha \lambda n^2}{2L} + O\left(\left(\frac{1}{L}\right)^2\right)$$

In the second part of the exercises we compare this result with the Yang-Mills prediction.

Weak coupling

• For given L and M the Bethe ansatz equation has many different solutions. They all correspond to different operators and different classical string. To find the solution which correspond to the circular string from the part 1 of this excessive we have to specify a good set of starting points.
Define the function StartingPoints[L_,M_] as follows

• Find numerically a solution of the Bethe ansatz equation using FindRoot function. Take L and M small.

According to the AdS/CFT duality the objects dual to the classical string energies at weak coupling are the anomalous dimensions. They are defined by two point functions:

$$\langle O_1(x)\bar{O}_1(x)\rangle \sim \frac{1}{(x-y)^{2\Delta}} .$$

where Δ can be found order by order in perturbation theory. The operators dual to the string moving in S^3 subspace are the scalar single trace operators of the form $O_1 = \mathrm{tr}(XZXXZ\dots) + \text{permutations}$. To the leading order in λ it is very easy to compute $\Delta_0 = \#X + \#Z \equiv L$ (see another exercises where this quantity is computed to the leading order). To the next order the problem can be solved by means of the Bethe ansatz

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = -\prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

where $M = \#X$. When the Bethe ansatz is solved the anomalous dimension of the operator is given by

$$\Delta = L + \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + 1/4} + \mathcal{O}(\lambda^2) .$$

```
StartingPoints[L_, M_, n_] := L/(2 π n) + I (Sqrt[2 L]/(2 π n)) zk /. NSolve[HermiteH[M, zk] == 0, zk];
```

• Define a function SolveBAE[L_,M_,n_] which solve the Bethe ansatz using these starting points. Check

```
SolveBAE[10, 2, 1]

{u(1) → 1.3088741844046434659894738897 3 − 0.58168645590651625751431799321 8 i,
  u(2) → 1.3088741844046434659894738897 3 + 0.58168645590651625751431799321 8 i}
```

• Plot the points generated by SolveBAE[210,30,1] on the complex plain (use ListPlot). You should see that the Bethe roots u_k are distributed along nice cuts

• Compute energy for the same solution SolveBAE[210,30,1]. You should get:

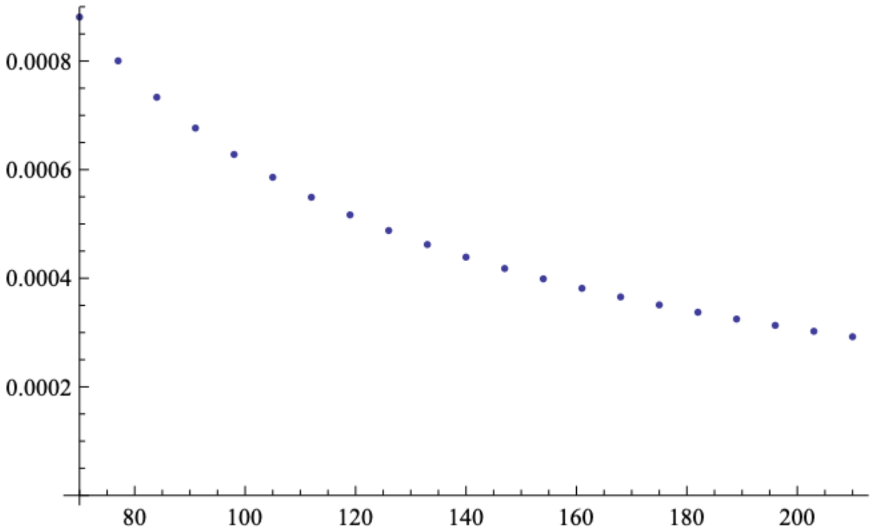
```
0.000292224974357119224624961305060 λ + 210
```

Compare this with the energy of the circular string from part1 of the exercises under identification $\alpha = M/L$.

• Make a list of coefficients in front of λ for $L = 7M$ and $M = 10 \ldots 30$ and call it `deltas`. it should be of the form `{{L,coefficient},...}` . Plot `deltas` to get

You should see that the results are very similar!!!
To make the test of AdS/CFT even more convincing we should extrapolate the result to large L.

```
ListPlot[deltas]
```



• Using Fit function make of fit of this data by many inverse powers of L. Try to match the coefficient $1/L$ with the AdS/CFT prediction $\alpha(1-\alpha)/2$ with $\alpha = 1/7$. It is possible to get at least 10 digits match!