Witten diagrams & Mellin Transform in AdS/CFT.

AdS/CFT duality

- Quantum Gravity on anti-de Sitter Spacetime (AdS_{d+1})
- d-dim conformal field theory (CFT) on boundary of AdS_{d+1}.

holographic principle

Well-known examples.

* Type IIB superstring on $AdS_5 \times S^5$
* 4d super Yang-Mills theory with $\mathcal{N}=4$ susy.
* M-theory on $AdS_4 \times S^7$
* 3d supersymmetric Chern-Simons matter theory (ABJM theory)
* M-theory on $AdS_7 \times S^4$
* 6d SCFT w/ (2,0) susy.

weak-strong duality

Basic aspects of CFT & AdS.

Conformal Symmetry. $x \to x'$

$$\delta_{\mu\nu} \rightsquigarrow C(x)\,\delta_{\mu\nu} = g_{\mu\nu}(x')$$

- Poincaré symmetry $(M_{\mu\nu}, P_\mu)$
- Dilatation (D): $x'_\mu = \lambda x_\mu$.
- Special Conformal Symmetry (K_μ)

$$x'_\mu = \frac{x_\mu - a_\mu x^2}{1 - 2(a\cdot x) + a^2 x^2}.$$

- Inversion $x'_\mu = \frac{x_\mu}{x^2}$, $\quad x'_\mu = x_\mu - a_\mu$. ⊕ Inversion

$$[D, K_\mu] = -K_\mu, \quad [D, P_\mu] = P_\mu,$$

$$[K_\mu, P_\nu] = \delta_{\mu\nu} D - 2i M_{\mu\nu}, \quad \cdots$$

- State-operator correspondence.

$$\mathcal{O}(x) \leftrightsquigarrow |\mathcal{O}\rangle = \mathcal{O}(0)\,\underbrace{|0\rangle}_{\text{Vacumm.}}$$

$$D|\mathcal{O}_\Delta\rangle = \underbrace{\Delta}_{\text{Conformal dim.}}|\mathcal{O}_\Delta\rangle, \quad [D, \mathcal{O}_\Delta(0)] = \Delta\,\mathcal{O}_\Delta(0)$$

Two Types: Primary, descendant

Primary: Cannot be written as derivatives acting other operatos

descendant: derivatives acting on other operators.

$$\begin{cases} K_\mu |\mathcal{O}_\Delta\rangle = 0 & [K_\mu, \mathcal{O}_\Delta(0)] = 0 \\ D|\mathcal{O}_\Delta\rangle = \Delta|\mathcal{O}_\Delta\rangle \end{cases}$$

$$\begin{cases} K_\mu : \text{annihilation} \\ P_\mu : \text{creation.} \end{cases}$$

$$\begin{cases} 0 \xleftarrow{K_\mu} \mathcal{O}_\Delta \xleftarrow{K_\mu} \mathcal{O}_{\Delta+1} \xleftarrow{K_\mu} \mathcal{O}_{\Delta+2} \cdots \\ \mathcal{O}_\Delta \xrightarrow{P_\mu} \mathcal{O}_{\Delta+1} \xrightarrow{P_\mu} \mathcal{O}_{\Delta+2} \cdots \end{cases}$$

$P_\mu \to \partial_\mu$

- Conformal symmetry fixes 2 & 3-pt correlation functions of Primary operators (scalar)

$$\langle \mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(x_{12}^2)^{\Delta_1}} \qquad x_{12}^\mu = x_1^\mu - x_2^\mu$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2)\,\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}(g, N \cdots)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}.$$

$$\langle \mathcal{O}_\Delta(x_1)\,\mathcal{O}_\Delta(x_2)\,\mathcal{O}_\Delta(x_3)\,\mathcal{O}_\Delta(x_4)\rangle = \frac{1}{(x_{13}^2)^\Delta (x_{24}^2)^\Delta}\, g(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = u(2 \Leftrightarrow 4)$$

- d-dim conformal symmetry $\sim$ $SO(d+1,1)$

Lorentz of $\mathbb{R}^{d+1,1}$ $\leftarrow$ X^A, $A=0,\underbrace{1,2\cdots d}_{\mu},d+1$

$$J_{AB}=-i\left(X_A\frac{\partial}{\partial X^B}-X_B\frac{\partial}{\partial X^A}\right)$$

$$M_{\mu\nu}=J_{\mu\nu},\quad D=J_{0\,d+1},\quad K_\mu=J_{0\mu}+J_{d+1,\mu}$$

$$P_\mu=J_{0\mu}-J_{d+1\,\mu}.$$

Embedding formalism

Embed d-dim space in (d+2)-dim $\mathbb{R}^{d+1,1}$.

light rays $\quad P^2=0 \qquad P\sim\lambda P$

$$x^\mu=\frac{P^\mu}{P^+}\qquad P^+=P^0+P^{d+1}$$

$$\left\{P^0=\frac{1+x^2}{2},\quad P^\mu=x^\mu,\quad P^{d+1}=\frac{1-x^2}{2}\right\}.$$

- defined in Embedding space

$$O_\Delta(x)\rightsquigarrow O_\Delta(\lambda P)=\lambda^{-\Delta}O_\Delta(P)$$

- Conformal symmetry constraints become (d+2)-dim Lorentz & homogeneity condition.

$$\langle O_{\Delta_1}(P_1)\, O_{\Delta_2}(P_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(-2P_1\cdot P_2)^{\Delta_1}}$$

$$\langle O_{\Delta_1}(P_1)\, O_{\Delta_2}(P_2)\, O_{\Delta_3}(P_3)\rangle = \frac{C_{123}}{(-2P_1\cdot P_2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(-2P_1\cdot P_3)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(-2P_2\cdot P_3)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}$$

$$-2P_i\cdot P_j = x_{ij}^2$$

AdS$_{d+1}$ Spacetime.

It's a solution to Eistein Eqs. with negative cosmological constant. $(X^A,\ A=0,1,\cdots,d+1)$

Hyperboloid:

$$-(X^0)^2+(X^1)^2+\cdots+(X^{d+1})^2=-R^2$$

Embedding in $\mathbb{R}^{d+1,1}$. R: Radius. $\underline{R=1}$

$$X=\left\{X^0=\frac{1+x^2+z^2}{2z},\ X^\mu=\frac{x^\mu}{z},\ X^{d+1}=\frac{1-x^2-z^2}{2z}\right\}.$$

Poincaré coordinates. with.

$$ds^2=\frac{dz^2+\delta_{\mu\nu}dx^\mu dx^\nu}{z^2}.$$

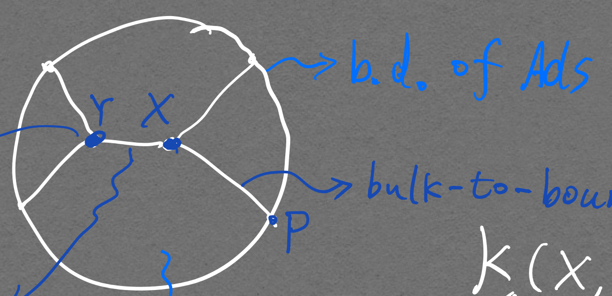
The conformal b.d. is at $z\to 0$. $P^2=0$

$$\underline{X} \rightsquigarrow \underline{\underline{P}} = \left\{P^0=\frac{1+x^2}{2},\ P^\mu=x^\mu,\ P^{d+1}=\frac{1-x^2}{2}\right\}$$

- AdS_{d+1} has isometry $\underline{SO(d+1,1)}$, which is precisely the conformal symmetry in $\mathbb{R}^d$.

Witten diagrams

They are Feynman diagrams in AdS.



Interactions.

$$S = \int_{AdS} dX \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \cdots \right]$$

$\lambda\phi^3, \lambda\phi^4, \ldots$

- bulk-to-bulk progator

$$(\nabla_X^2 - m^2) G_\Delta(X,Y) = -\delta(X,Y)$$

$$\nabla_X^2 = -X^2\partial_X^2 + X\cdot\partial_X\left[d + X\cdot\partial_X\right]$$

$$\boxed{G_\Delta(X,Y) = \frac{C_\Delta}{u^\Delta}\, {}_2F_1\left(\Delta, \Delta - h + \tfrac{1}{2}, 2\Delta - 2h + 1, -\frac{4}{u}\right)}$$

$u = (X-Y)^2$, $\quad h = d/2$, $\quad C_\Delta = \dfrac{\Gamma(\Delta)}{2\pi^h\Gamma(\Delta - h + 1)}$

$$Y = \lambda P + \underset{\substack{\downarrow \\ \text{do not grow } \lambda\to\infty}}{\cdots} \qquad \underline{Y^2=-1}$$

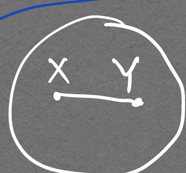
$u \to \infty$

$$K_\Delta(X,P) = \lim_{\lambda\to\infty} \lambda^\Delta G_\Delta(X, \lambda P + \cdots)$$

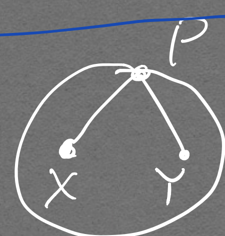
$$= \frac{C_\Delta}{(-2P\cdot X)^\Delta}$$

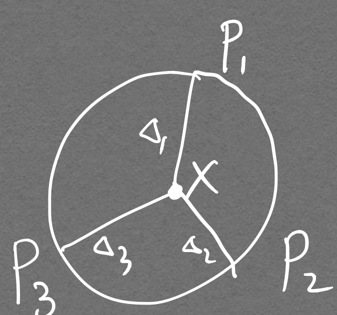
$$= \frac{1}{2\pi^h \Gamma(1+\Delta-h)} \int_0^\infty \frac{dt}{t} t^\Delta e^{2tP\cdot X}$$

$$\bullet \quad G_\Delta(X,Y) = \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \underbrace{\frac{2c^2}{c^2-(\Delta-h)^2}}_{g_\Delta(c)} \int_{\partial AdS} dP\, K_{h+c}(X,P) K_{h-c}(Y,P)$$



$$= \int \frac{dc}{2\pi i} g_\Delta(c) \int_{\partial AdS} dP$$





$\underline{\lambda\phi^3}$

$$A(P_1,P_2,P_3) = \lambda \int_{AdS} dX\, K_{\Delta_1}(X,P_1) K_{\Delta_2}(X,P_2) K_{\Delta_3}(X,P_3)$$

$$= \prod_{i=1}^{3} \frac{1}{2\pi^h \Gamma(1+\Delta_i-h)} \int_{AdS} dX \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{\Delta_i} e^{2(\sum_i t_i P_i)\cdot X}$$

use:

(exercise) $$\boxed{\int_{AdS} dX\, e^{2Q\cdot X} = \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+Q^2/z}}$$

$$= \pi^h \prod_{i=1}^{3} \frac{1}{2\pi^h \Gamma(1+\Delta_i-h)} \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{\Delta_i} \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+\frac{2\sum_{i<j} t_i t_j P_i\cdot P_j}{z}}$$

$t_i \to t_i\sqrt{z}$

$$= \frac{\pi^h \Gamma(\frac{\sum\Delta_i}{2}-h)}{\prod_{i=1}^{3} 2\pi^h \Gamma(1+\Delta_i-h)} \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{\Delta_i} e^{2\sum_{i<j} t_i t_j P_i\cdot P_j}$$

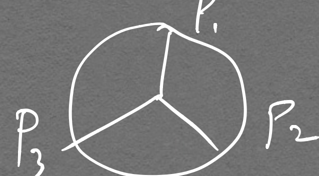
use Symanzik "star formula" (exercise)

$$\boxed{\int_0^\infty \prod_{i=1}^{n} \frac{dt_i}{t_i} t_i^{\Delta_i} e^{2\sum_{i<j} t_i t_j P_i\cdot P_j} = \frac{1}{2}\int_{-i\infty+c}^{i\infty+c} \prod_{i<j} \frac{d\gamma_{ij}}{2\pi i} \Gamma(\gamma_{ij}) (-2P_i\cdot P_j)^{-\gamma_{ij}}}$$

$\gamma_{ij} = \gamma_{ji}$, $\sum_{j\neq i} \gamma_{ij} = \Delta_i$,

$$= \frac{\pi^h \Gamma(\frac{\sum\Delta_i}{2}-h)}{\prod_{i=1}^{3} 2\pi^h \Gamma(1+\Delta_i-h)} \frac{1}{2} \int \prod_{i<j}^{3} \frac{d\gamma_{ij}}{2\pi i} \Gamma(\gamma_{ij}) (-2P_i\cdot P_j)^{\gamma_{ij}} \times 1$$

$= A(P_1, P_2, P_3)$



λ.

1). $\sum_{j\neq i} \gamma_{ij} = \Delta_i$, $\Rightarrow \begin{cases} \gamma_{12} + \gamma_{13} = \Delta_1 \\ \gamma_{12} + \gamma_{23} = \Delta_2 \\ \gamma_{13} + \gamma_{23} = \Delta_3 \end{cases}$

$$\Rightarrow \gamma_{12} = \frac{\Delta_1+\Delta_2-\Delta_3}{2}, \quad \gamma_{13} = \frac{\Delta_1+\Delta_3-\Delta_2}{2}, \quad \gamma_{23} = \frac{\Delta_2+\Delta_3-\Delta_1}{2}.$$

$$A(P_1,P_2,P_3) \sim (-2P_1\cdot P_2)^{-\frac{\Delta_1+\Delta_2-\Delta_3}{2}} (-2P_1\cdot P_3)^{-\frac{\Delta_1+\Delta_3-\Delta_2}{2}} (-2P_2\cdot P_3)^{-\frac{\Delta_2+\Delta_3-\Delta_1}{2}}$$

2) The 3-pt result is naturally in Mellin Space.

In general, n-point

$$\langle O_{\Delta_1}(P_1) O_{\Delta_2}(P_2) \cdots O_{\Delta_n}(P_n) \rangle$$

$$= \int \prod_{i<j} \frac{d\gamma_{ij}}{2\pi i} (-2P_i\cdot P_j)^{-\gamma_{ij}} \underline{\Gamma(\gamma_{ij})}\ \underline{M(\gamma_{ij})}$$

$$\gamma_{ij} = \gamma_{ji}, \quad \sum_{j\neq i} \gamma_{ij} = \Delta_i$$

$$\Rightarrow \gamma_{ij} = k_i\cdot k_j, \quad \underline{\sum_{i=1}^{n} k_i^\mu = 0 \qquad k_i^2 = -\Delta_i}$$

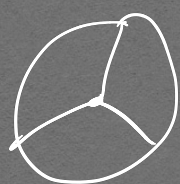
k_i plays the role of momentum.

γ_{ij} " " " " Mandanstem Varibles.

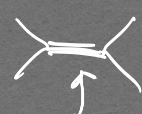
$M(\gamma_{ij})$ is <u>Mellin amplitude</u>.

• Take Radius of AdS to $\underline{\infty}$.

$M(\gamma_{ij})$ reproduces the flat-space Amplitudes.

 : $M(\gamma_{ij}) \sim \lambda$ $\qquad \lambda\phi^3$

• Operator product expansion OPE

: $\rightsquigarrow$ 

OPE singularity is reflected in poles in Mellin space. $\underline{\Gamma(\gamma_{ij})}$

poles in $\Gamma(\gamma_{ij})$ reprents double-trace operators.

poles in $\underline{M(\gamma_{ij})}$ " single-trace operators.