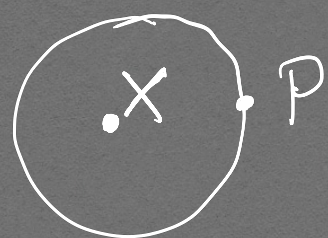
Witten diagrams & Mellin Transform in AdS/CFT

• Integral Identities: $R^2=-1$

$z>0$



$\mathbb{R}^{1,d+1}$

$$\begin{cases} X = \left\{ \dfrac{1+x^2+z^2}{2z},\ \underset{\text{d-dim}}{\dfrac{x^\mu}{z}},\ \dfrac{1-x^2-z^2}{2z} \right\} \\ Q = |Q|\{1, \vec{0}, 0\} \end{cases}$$

$Q^2=-|Q|^2$

$$\frac{-dz^2+\delta_{\mu\nu}dx^\mu dx^\nu}{z^2}=ds^2$$

$$\int_{AdS_{d+1}} dX\, e^{2Q\cdot X}$$

$$= \int_0^\infty \frac{dz}{z}\int \frac{d^d\vec{x}}{z^d}\, e^{-|Q|\frac{1+x^2+z^2}{z}}$$

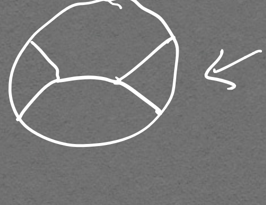
$$\int dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

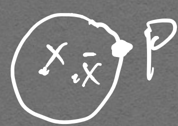
$h=d/2.$

$$= \int_0^\infty \frac{dz}{z^{d+1}}\, e^{-\frac{1+z^2}{z}|Q|}\left(\frac{\pi}{|Q|/z}\right)^h$$

$$= \pi^h |Q|^{-h} K_h(2|Q|) = \pi^h\left(\sqrt{-Q^2}\right)^{-h} K_h\left(2\sqrt{-Q^2}\right)$$

$$= \pi^h \int_0^\infty \frac{dz}{z}\, z^{-h}\, e^{-z+Q^2/z}$$



- 

$P^2=0 \qquad P=\left\{\frac{1+x^2}{2},\, x^\mu,\, \frac{1-x^2}{2}\right\}$

$$\int_0^\infty \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\int_{\partial AdS} dP\, e^{2P\cdot(sX+\bar{s}\bar{X})}$$

$|sX+\bar{s}\bar{X}|\{1,\vec{0},0\}$

$$\boxed{= 2\pi^h \int_0^\infty \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\, e^{(sX+\bar{s}\bar{X})^2}}$$

$$=\int_0^\infty \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\int d^dx\, e^{-(1+x^2)|sX+\bar{s}\bar{X}|}$$

$$=\int_0^\infty dv\int_0^\infty \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\,\frac{\pi^h}{|sX+\bar{s}\bar{X}|^h}\, e^{-|sX+\bar{s}\bar{X}|}\,\delta(v-s-\bar{s})$$

$s\to sv,\ \bar{s}\to\bar{s}v$

$$=\int_0^\infty \frac{dv}{v}\int_0^\infty \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\, v^h\,\frac{\pi^h}{|sX+\bar{s}\bar{X}|^h}\, e^{-v|sX+\bar{s}\bar{X}|}\,\delta(1-s-\bar{s})$$

$$=\int_0^\infty \frac{dv}{v}\int \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\, v^h\,\pi^h\, e^{v(sX+\bar{s}\bar{X})^2}\,\delta(1-s-\bar{s})$$

$s\to s/\sqrt{v}$

$\bar{s}\to \bar{s}/\sqrt{v}$

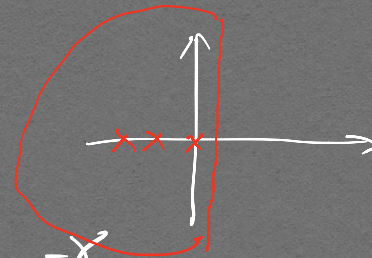
$$=\int_0^\infty \frac{dv}{v}\int \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, s^{h+c}\,\bar{s}^{h-c}\,\pi^h\, e^{(sX+\bar{s}\bar{X})^2}\,\delta\!\left(1-\frac{s}{\sqrt{v}}-\frac{\bar{s}}{\sqrt{v}}\right)$$

- Symanzik "star formula"

$$\int_0^\infty \prod_{i=1}^n \frac{dt_i}{t_i}\, t_i^{\Delta_i}\, e^{2\sum_{i<j} t_i t_j P_i\cdot P_j}$$

$$=\frac{1}{2}\int_{-i\infty+c}^{+i\infty+c}\prod_{i<j}\frac{d\gamma_{ij}}{2\pi i}\,\Gamma(\gamma_{ij})\,(-2P_i\cdot P_j)^{-\gamma_{ij}}$$

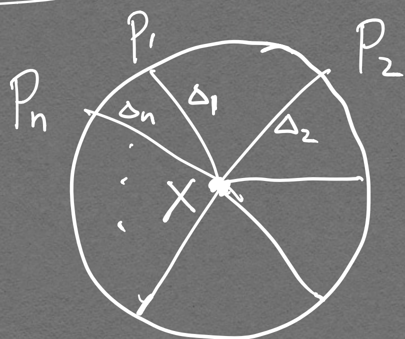
$$\sum_{j\neq i} \gamma_{ij} = \Delta_i \,, \quad \gamma_{ij} = \gamma_{ji}$$



using $$e^{x} = \int_{-i\infty+c}^{i\infty+c} \frac{d\gamma}{2\pi i}\,\Gamma(\gamma)\,(-x)^{-\gamma}$$

$$= 1 + x + \frac{x^2}{2!} + \cdots$$

do the $\int dt_i$, then arrive at the identity.



$$\lambda\, \phi_1 \phi_2 \cdots \phi_n \,.$$

$$K_\Delta(X,P) = \frac{C_\Delta}{(-2X\cdot P)^\Delta} = \frac{1}{2\pi^h\,\Gamma(1+\Delta-h)} \int \frac{dt}{t}\, t^\Delta e^{2P\cdot X}$$

$$= D_{\Delta_1 \Delta_2 \cdots \Delta_n}(P_1 \cdots P_n)$$

$$A_n(P_1,\cdots,P_n) = \lambda \int_{AdS_{d+1}} dX\, K_{\Delta_1}(X,P_1)\, K_{\Delta_2}(X,P_2) \cdots K_{\Delta_n}(X,P_n)$$

$$= \frac{\lambda}{\prod_{i=1}^{n} 2\pi^h\,\Gamma(1+\Delta_i-h)} \int_{AdS_{d+1}} dX \prod_{i=1}^{n} \frac{dt_i}{t_i}\, t_i^{\Delta_i}\, e^{2X\cdot\left(\sum_{i=1}^{n} t_i P_i\right)}$$

$$= \frac{\lambda}{\prod_{i=1}^{n} 2\pi^h\,\Gamma(1+\Delta_i-h)} \int \prod_{i=1}^{n} \frac{dt_i}{t_i}\, t_i^{\Delta_i}\, e^{2\sum_{i<j} t_i t_j\, P_i\cdot P_j}$$

$$= \frac{\lambda}{\prod_{i=1}^{n} 2\pi^h\,\Gamma(1+\Delta_i-h)}\, \frac{1}{2} \int \prod_{i<j} \frac{d\gamma_{ij}}{2\pi i} \overset{\rightsquigarrow\ \frac{n(n-3)}{2}}{\underset{\delta\left(\sum_j \gamma_{ij} - \Delta_i\right)}{\Gamma(\gamma_{ij})}}\, (-2P_i\cdot P_j)^{-\gamma_{ij}} \;\underline{\times 1}$$

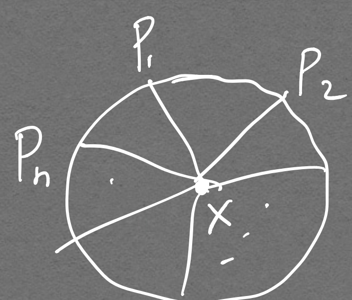
$$M_n(\gamma_{ij}) \sim \lambda \qquad \sum_{j\neq i}\gamma_{ij} = \Delta_i \qquad O_\Delta(\lambda P) = \lambda^{-\Delta} O_\Delta(P)$$

$$P_i \to \lambda_i P_i \qquad A_n \sim \lambda_i^{-\sum_{j\neq i}\gamma_{ij}} = \lambda_i^{-\Delta_i}$$

- $n=3$. $\int d\gamma_{12}\, d\gamma_{13}\, d\gamma_{23}$

$$\left.\begin{aligned} \gamma_{12}+\gamma_{13} &= \Delta_1 \\ \gamma_{12}+\gamma_{23} &= \Delta_2 \\ \gamma_{13}+\gamma_{23} &= \Delta_3 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \gamma_{12} &= \frac{\Delta_1+\Delta_2-\Delta_3}{2} \\ \gamma_{13} &= \frac{\Delta_1+\Delta_3-\Delta_2}{2} \\ \gamma_{23} &= \frac{\Delta_2+\Delta_3-\Delta_1}{2} \end{aligned}\right.$$

$$A_3 \sim \frac{1}{(-2P_1\cdot P_2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(-2P_1\cdot P_3)^{\bigcirc}(-2P_2\cdot P_3)^{\Delta}}$$

- 

$$\lambda \nabla\phi_1\cdot\nabla\phi_2\, \phi_3 \cdots \phi_n .$$

$$\nabla_A \phi = U_A^{\ B} \frac{\partial \phi}{\partial X^B} = (\eta_A^{\ B} + X^B X_A)\frac{\partial\phi}{\partial X^B}.$$

$$A_n = \int_{AdS} dX\, \nabla K_{\Delta_1}(X,P_1)\cdot\nabla K_{\Delta_2}(X,P_2)\, K_{\Delta_3}\cdots K_{\Delta_n}$$

- $$\nabla K_{\Delta_1}\cdot\nabla K_{\Delta_2} = \nabla\frac{C_{\Delta_1}}{(-2P_1\cdot X)^{\Delta_1}}\cdot\nabla\frac{C_{\Delta_2}}{(-2P_2\cdot X)^{\Delta_2}}$$

$P_1\cdot P_2$

$$= \frac{C_{\Delta_1}C_{\Delta_2}\,(P_1\cdot P_2 + P_1\cdot X\, P_2\cdot X)}{(-2P_1\cdot X)^{\Delta_1+1}(-2P_2\cdot X)^{\Delta_2+1}}$$

$$= P_1\cdot P_2 \int_{AdS} dX\, K_{\Delta_1+1}(X,P_1)\, K_{\Delta_2+1}(X,P_2)\, K_{\Delta_3}\cdots K_{\Delta_n}$$

$$= P_1\cdot P_2 \int \frac{d\gamma_{ij}}{2\pi i}\, \Gamma(\gamma_{ij})\, (-2P_i\cdot P_j)^{-\gamma_{ij}}$$

$$\sum_{j\neq 1}\gamma_{1j} = \Delta_1+1, \quad \sum_{j\neq 2}\gamma_{2j} = \Delta_2+1, \quad \sum_{j\neq i}\gamma_{ij} = \Delta_i$$

$$\to \gamma_{12} = \gamma_{12}'+1 \qquad \gamma_{ij} = \gamma_{ij}'$$

$$= \int \frac{d\gamma_{ij}}{2\pi i}\, \Gamma(\gamma_{ij})\, (-2P_i\cdot P_j)^{-\gamma_{ij}} \times \gamma_{12}$$

2-derivatives

$$\lambda \nabla\phi_1\cdot\nabla\phi_2\phi_3\cdots\phi_n \qquad \sum_{j\neq i}\gamma_{ij} = \Delta_i$$

$$M_n(\gamma_{ij}) = {}_{\#}\overbrace{\gamma_{12} + {}_{\#}1}^{s_{12}}$$

2m-derivatives contact interaction.

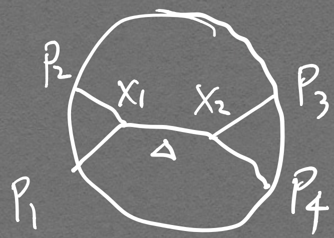
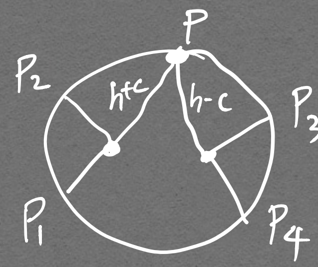
$M_n^{(m)}(\gamma_{ij})$ is a degree-m Polynomial in γ_{ij}.

- Flat-space Limit $R\to\infty$, $\gamma_{ij}\to\infty$.

$$M_n(\gamma_{ij}) \xrightarrow{\text{flat-space}} T_n(s_{ij})$$

$\downarrow$

flat-space Amplitude

•  $= \int_{\partial AdS} dP \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i}\, g_\Delta(c)$ 

$$= \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i}\, g_\Delta(c) \int_{\partial AdS} dP\, A_3^{(+)}(P_1 P_2 P)\, A_3^{(-)}(P P_3 P_4)$$

$$g_\Delta(c) = \frac{2c^2}{c^2-(\Delta-h)^2}$$

plug in A_3, $\int_{\partial AdS} dP$

$$= 2\pi^{3h} \int \frac{dc}{2\pi i} f_\Delta(c) \int \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, \Gamma\left(\frac{\Delta_1+\Delta_2+c-h}{2}\right) \Gamma\left(\frac{\Delta_3+\Delta_4-c-h}{2}\right) s^{h+c}\, \bar{s}^{-h-c}$$

$$\int \frac{dt_i}{t_i}\, t_i^{\Delta_i} \exp\Big\{-(1+s^2)\, t_1 t_2 P_{12} - (1+\bar{s}^2)\, t_3 t_4 P_{34} - s\bar{s} \sum_{(i,j)} t_i t_j P_{ij}\Big\}$$

$$\boxed{P_{ij} = -2P_i\cdot P_j}$$

"star formula"

$(i,j) \rightarrow (1,3),\ (1,4),\ (2,3),\ (2,4)$

$$= 2\pi^{3h} \int \frac{dc}{2\pi i} f_\Delta(c) \int \frac{ds}{s}\frac{d\bar{s}}{\bar{s}}\, \Gamma\left(\frac{\Delta_1+\Delta_2+c-h}{2}\right) \Gamma\left(\frac{\Delta_3+\Delta_4-c-h}{2}\right) s^{h+c}\, \bar{s}^{-h-c}$$

$$\int \frac{d\gamma_{ij}}{2\pi i} (1+s^2)^{-\gamma_{12}} (1+\bar{s}^2)^{-\gamma_{34}}\, s^{h+c-\sum_{(i,j)}\gamma_{ij}}\, \bar{s}^{h-c-\sum_{(i,j)}\gamma_{ij}}\, \Gamma(\gamma_{ij})\, P_{ij}^{-\gamma_{ij}}$$

$\int ds$ & $\int d\bar{s} \rightarrow$ ratios of Γ-functions

$$= 2\pi^{3h} \int \frac{dc}{2\pi i} f_\Delta(c)\, \Gamma\left(\frac{\Delta_1+\Delta_2+c-h}{2}\right) \Gamma\left(\frac{\Delta_3+\Delta_4-c-h}{2}\right)$$

$$\int \frac{d\gamma_{ij}}{2\pi i} \frac{\Gamma\left(\gamma_{12} - \frac{h+c-\sum_{(i,j)}\gamma_{ij}}{2}\right)\Gamma\left(\frac{h+c-\sum_{(i,j)}\gamma_{ij}}{2}\right)}{2\Gamma(\gamma_{12})} \times \left(\gamma_{12} \rightarrow \gamma_{34}\right)$$

$$f_\Delta(c) = \frac{1}{2\pi^{2h}\left[(\Delta-h)^2-c\right)} \frac{1}{\Gamma(c)\Gamma(-c)}$$

$$\boxed{\begin{array}{l} \gamma_{ij} = k_i \cdot k_j \\ \sum k_i^\mu = 0 \\ k_i^2 = -\Delta_i \end{array}}$$

$$\int dc \rightsquigarrow {}_3F_2 \qquad S_{12} = -(k_1+k_2)^2 = \Delta_1+\Delta_2-2\gamma_{12}$$

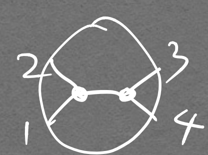
$$M_4(S_{12}) = \frac{\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta-h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+\Delta-h}{2}\right)}{2\,\Gamma(1+\Delta-h)} \frac{1}{S_{12}-\Delta}$$

$${}_3F_2\left(\frac{2-\Delta_1-\Delta_2+\Delta}{2}, \frac{2-\Delta_3-\Delta_4+\Delta}{2}, \frac{\Delta-S_{12}}{2}; \frac{2+\Delta-S_{12}}{2}, 1+\Delta-h; 1\right)$$

$$\Delta_i = \Delta = d = 4 \qquad R^2 m^2 = \Delta(\Delta-d)$$

$$M_4(S_{12}) = 48\left(\frac{1}{S_{12}-6} + \frac{1}{S_{12}-4}\right)$$

$$M_4(S_{12}) = \sum_{m=0}^{\infty} \frac{P_m^\Delta}{S-\Delta-2m} V^{\Delta_1\Delta_2\Delta}_{[0,0,m]} V^{\Delta_3\Delta_4\Delta}_{[0,0,m]}$$



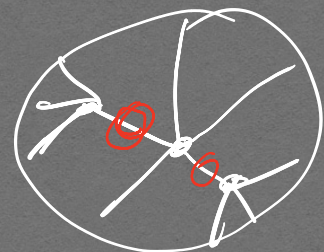
$$P_m^\Delta = \frac{1}{2m!\,\Gamma(1+\Delta-h+m)},$$

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

$$V^{\Delta_1\Delta_2\Delta_3}_{[0,0,m]} = V^{\Delta_1\Delta_2\Delta_3}_{[0,0,0]} \left(1 - \frac{\sum\Delta_i}{2} + \Delta_3\right)_m$$

$$V^{\Delta_1\Delta_2\Delta_3}_{[0,0,0]} = \Gamma\left(\frac{\sum\Delta_i - h}{2}\right)$$

$m=0$, Primary, $m>0$, descendants.



Feynman rules

$\gamma_{ij} \to \infty$

$\lambda \phi^4$

Recent developments.

- generalize to Spins, Loops
- generalize dS, relevant to cosmoly observables.

— 4d $\mathcal{N}=4$ SYM $\Longleftrightarrow$ IIB string on $AdS_5 \times S^5$.

$R \to \infty$. in $\mathbb{R}^{9,1}$

* Bootstrap. using:

* pole structures, symmetry constraints. power counting. & other tools.

AdS correlators without Witten diagrams.

— 6d (2,0) SCFT $\Longleftrightarrow$ M-theory on $AdS_7 \times S^4$ $\mathbb{R}^{10,1}$

3d ABJM theory $\Longleftrightarrow$ M-theory on $AdS_4 \times S^7$.

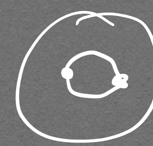
Half-maximal SUSY

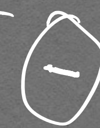
2d SCFT $\Longleftrightarrow$ $AdS_3 \times S^3$

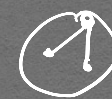
$\boxed{\text{Loops}}$

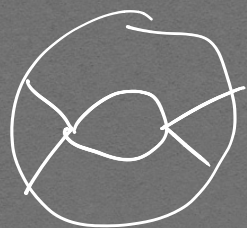
EFT

ϕ^n



$= \sum$ 

$= \sum$ 



ϕ^4

$= \sum$