London Theory Institute Lectures Series

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“ Entanglement in 1+1D Quantum Field Theory ”

Informations

Pre-recorded Lectures : Youtube
Live Tutorial : Monday 22nd of March, 11h00

Abstract

In this short course I will introduce branch point twist fields and explain how they emerge in the context of computing entanglement measures in 1+1D. I will focus on massive 1+1D integrable quantum field theory (IQFT) and also comment on some well-known results in conformal field theory (CFT).

The talk will be structured into three main parts:

First, I will introduce entanglement measures, focussing on the entanglement entropy, explain how these measures relate to partition functions in multi-sheeted Riemann surfaces and how these, in turn, may be expressed as correlators of branch point twist fields.

Second, I will show how several well-known results in CFT and IQFT are very easily derived in this branch point twist field picture and how they can also be recovered numerically in a quantum spin chain.

Finally, I will explain how more involved computations with branch point twist fields may be performed by exploiting form factor technology and will end the talk by showing an example of one such calculation.

Entanglement Measures and Branch Point Twist Fields

EXERCISES

1. Given that the entropy of region of length ℓ with two boundary points in an infinite unitary critical system diverges as:

$$S(\ell) = \frac{c}{3} \log \frac{\ell}{\varepsilon},$$

where ε is a non-universal short-distance cut-off, use conformal maps to find the equivalent expressions for the entanglement entropy of sub-system of length ℓ with two boundary points within a finite system of length L. Hint: You just need to find the appropriate conformal map.

2. The branch point twist field and its conjugate are primary fields in a replica conformal field theory of central charge nc, where c is the central charge of each replica. Compute the expectation value of the stress-energy tensor of the replica CFT in the n-sheeted manifold seen in the lecture, that is $\langle T(w)\rangle_{\mathcal{M}_{n,x_1,x_2}}$, where x_1, x_2 are the branch points. Hint: Again you need to find the appropriate conformal map and use the transformation law of the stress-energy tensor in CFT.

3. We have seen that the replica partition function of a finite connected interval is proportional to the two-point function of branch point twist fields $\langle \mathcal{T}(x_1)\mathcal{T}^\dagger(x_2)\rangle = |x_1 - x_2|^{-4\Delta_{\mathcal{T}}}$. From this and the result of the previous question, find the conformal dimension of the branch point twist fields $\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right)$. Note: My definition of $\Delta_{\mathcal{T}}$ is unusual. Most people give this name to twice the same quantity, so watch out!

4. Consider the following CFT results for two-, three- and four-point functions of primary fields:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = |x_1 - x_2|^{-4\Delta},$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\mathcal{C}_{123}}{|x_1 - x_2|^{2\Delta_1+2\Delta_2-2\Delta_3}|x_2 - x_3|^{2\Delta_2+2\Delta_3-2\Delta_1}|x_1 - x_3|^{2\Delta_1+2\Delta_3-2\Delta_2}},$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{F}(x)\left[\frac{|x_1 - x_3||x_2 - x_4|}{|x_1 - x_2||x_3 - x_4||x_1 - x_4||x_2 - x_3|}\right]^{4\Delta}$$

the last equation is true when the dimensions of all four fields are the same Δ and $\mathcal{F}(x)$ is a model dependent function of the cross-ratio

$$x = \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}.$$

Use the first two formulae to obtain the formulae for the Rényi and von Neumann entropies of a finite interval and for the (replica) logarithmic negativity of adjacent regions. For free fermions the function $\mathcal{F}(x) = 1$. In this case, find the Rényi and entanglement entropies of two disconnected regions.

5. Going back to the lecture notes, show that if the S-matrix has an integral representation of the form,

$$S_{ab}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} f_{ab}(t) \sin\frac{\theta t}{i\pi}\right],$$

then the function

$$F_{\textbf{min}}^{\mathcal{T}|ab}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\frac{f_{ab}(t)}{\sinh(nt)} \sin^2\left(\frac{it}{2}\left(n+\frac{i\theta}{\pi}\right)\right)\right]$$

satisfies

$$F_{\textbf{min}}^{\mathcal{T}|ab}(\theta) = S_{ab}(\theta) F_{\textbf{min}}^{\mathcal{T}|ba}(-\theta) = F_{\textbf{min}}^{\mathcal{T}|ba}(-\theta + 2\pi i n),$$

Assume parity invariance, that is $S_{ab}(\theta) = S_{ba}(\theta)$.

6. Show that the two-particle form factor of particles in the same copy

$$F_2^{\mathcal{T}|ab}(\theta) = \frac{\langle \mathcal{T}\rangle \sin\frac{\pi}{n}}{2n \sinh\left(\frac{i\pi-\theta}{2n}\right)\sinh\left(\frac{i\pi+\theta}{2n}\right)} \frac{F_{\textbf{min}}^{\mathcal{T}|ab}(\theta)}{F_{\textbf{min}}^{\mathcal{T}|ab}(i\pi)}$$

satisfies Watson's equations as well as the (first) kinematic residue equation:

$$-i \lim_{\theta\to i\pi} (\theta - i\pi) F_2^{\mathcal{T}|ab}(\theta) = F_0^{\mathcal{T}} := \langle \mathcal{T}\rangle$$

7. By computing the analytic continuation in n of $h(\theta, n)$, compute

$$\lim_{n\to 1}\frac{\partial h}{\partial n} \quad \text{with} \quad h(\theta, n) := \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}(\theta, n)$$

where $h_{ij}(\theta) := F_2^{\mathcal{T}|(a,i)(a,j)}(\theta)$. Check your result numerically for the free massive boson and fermion which have $F_{\textbf{min}}^{\mathcal{T}|11}(\theta) = 1$ and $F_{\textbf{min}}^{\mathcal{T}|11}(\theta) = -i\sinh\frac{\theta}{2n}$, respectively for particles in the same copy (note that there is a single particle species in these theories). Hint: Use the cotangent trick as we saw in lecture.

SOME MORE GENERAL QUESTIONS TO THINK ABOUT

1. Are any of the entanglement measures seen in the course experimentally measurable?

2. Why do we say that the entanglement entropy of gapped systems satisfies an area law?

3. Why do we say that the entanglement entropy of CFT violates the area law?

4. How do we know that the form factor expansion of correlation functions is rapidly convergent?

5. How would the equations for the two-particle form factor change if the theory is non-diagonal?

6. Are the solutions to the form factor equations unique? More precisely, is the two-particle solution of question 6 unique?

A Very Brief Bibliography

1. Integrable Quantum Field Theory

   Many ideas used in this course are basic ideas about integrable QFT. If you want to read more about this, there are some references that I particularly like:

   - A.B. Zamolodchikov, Integrable Field Theory from Conformal Field Theory, https://doi.org/10.2969/aspm/01910641
     This paper sees IQFTs as massive perturbations of CFT and shows how from this premise conserved quantities can be derived and classified and even new S-matrices constructed.
   - P.E. Dorey, Exact S-matrices, https://arxiv.org/abs/hep-th/9810026
     This is a very nice review of the bootstrap programme for 1+1D massive integrable QFT.

2. Branch Point Twist Fields

   - Branch point twist fields defined as symmetry fields associated with cyclic permutation symmetry in a replica theory were introduced in J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, https://arxiv.org/abs/0706.3384.
   - The first time the entanglement entropy was described in terms of the correlation function of a field was in the famous work of P. Calabrese and J.L. Cardy, Entanglement Entropy and Quantum Field Theory, https://arxiv.org/abs/hep-th/0405152. However, as I said in my lecture, their field Φ does not have the same conformal dimension as $\mathcal{T}$. In this work, the entanglement entropy of one interval is the two point function of Φ and its conjugate *raised to the power* n. So it is a different description.

3. Form Factors

   - Although the form factor equations are modified for the branch point twist field, the basic derivation is the same. A good place to read more about this, in a similar style as my course is the book of G. Mussardo, Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics, (Oxford Graduate Texts) 2009. If you cannot get the book, many of Mussardo's papers from the 90s cover similar ideas about form factors in integrable models. The same book is also a good reference for exact S-matrices and bootstrap programme in general.
   - A good review of the main properties can also be found in my PhD Thesis https://arxiv.org/abs/hep-th/0109212.
   - The form factor equations for branch point twist fields and the calculation of the exponential corrections appeared first in J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, https://arxiv.org/abs/0706.3384. We later wrote a review article covering the same ideas in a bit more detail: O.A. Castro-Alvaredo and B. Doyon, Bi-partite entanglement entropy in massive 1+1-dimensional quantum field theories, https://arxiv.org/abs/0906.2946.