What is an Anomaly?

Lectures for New PhD Students
London Theory Institute
Problem Set

due Monday 25 October

1. **Charged Bead on a Wire with a Magnetic Field:** Consider a bead of mass m and charge q constrained to move on a circular wire of radius R. Concentric with the wire, there is a constant magnetic field of strength B in a disk shaped region of radius $R_0 < R$. Show that the quantum mechanical Hamiltonian for the bead takes the form

$$H = c\left(-i\frac{\partial}{\partial\theta} + \gamma\right)^2 .$$

Express the constants c and γ in terms of m, q, B, R, and R_0.

The usual form for the Hamiltonian of a charged particle in the presence of a magnetic field is

$$H = \frac{(p - qA)^2}{2m}$$

where $\nabla \times A = B$. Rotational symmetry suggests that there is a gauge where A is purely tangential. From Stokes' Theorem, we have then for a disk shaped region of radius $R > R_0$,

$$\pi R_0^2 B = \int_{r<R} B \, d^2x = \int_{r<R} \nabla \times A \, d^2x = \oint_{r=R} A \cdot ds = 2\pi R|A|$$

In other words, $A_\theta = \frac{\Phi}{2\pi R}$ where $\pi R_0^2 B = \Phi$ is the magnetic flux.

Plugging into the Hamiltonian and using that $p_\theta = \frac{1}{iR}\partial_\theta$, we find that

$$H = \frac{(p_\theta - qA_\theta)^2}{2m} = \frac{1}{2mR^2}\left(-i\partial_\theta - \frac{q\Phi}{2\pi}\right)^2 .$$

2. **A Dihedral Group:** Show that the elements -1, R_π, and C, subject to the constraints $R_\pi^2 = \mathrm{id} = C^2$ and $CR_\pi C = -R_\pi$ generate the dihedral group D_8.

The dihedral group is generated by a 90° rotation a and a reflection x subject to the conjugation relation $xax = a^{-1}$. Also clearly $a^4 = 1 = x^2$. From the statement of the problem, we see that

$$CR_\pi CR_\pi = -1 ,$$

and hence that

$$(CR_\pi)^4 = 1$$

Suggesting we equate $a = CR_\pi$ in this 2d representation. We further hypothesize that $C = x$, and so we investigate

$$CCR_\pi C = R_\pi C .$$

Now note that $CR_\pi R_\pi C = 1$ and so indeed $R_\pi C = a^{-1}$.

3. **Weyl transformations:** Verify the following Weyl transformation rules:

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \delta^\lambda_\mu \partial_\nu \omega + \delta^\lambda_\nu \partial_\mu \omega - g_{\mu\nu}\partial^\lambda \omega \ , \\
\delta n_\mu &= \omega n_\mu \ , \quad \delta n^\mu = -\omega n^\mu \ , \\
\delta K_{\mu\nu} &= \omega K_{\mu\nu} + h_{\mu\nu} n^\lambda \partial_\lambda \omega \ , \\
\delta K &= h^{\mu\nu} K_{\mu\nu} = -\omega K + 2 n^\lambda \partial_\lambda \omega \ , \\
\delta R_{(2d)} &= -2\omega R_{(2d)} - 4\Box\omega \ , \quad \delta\Box_{(2d)} = -2\omega \Box_{(2d)} \ .
\end{aligned}$$

A Weyl transformation acts on the metric as $g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu}$. Here n_μ is a unit normal to the boundary, $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is a projector onto the boundary, and $K_{\mu\nu} = h_{\mu\lambda}\nabla^\lambda n_\nu$ is the extrinsic curvature.

4. **The Conformally Coupled Scalar:** Find the value of the constant ξ that makes the following scalar field action Weyl invariant:

$$S = -\frac{1}{2}\int d^d x \sqrt{\det g}\left[(\partial_\mu \phi)(\partial^\mu \phi) + \xi R \phi^2\right]$$

where R is the Ricci scalar curvature. Assume the Weyl transformation rules $g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu}$ and $\phi(x) \to e^{-\frac{d-2}{2}\omega(x)}\phi(x)$.

At linear order, the shift in the metric is $\delta g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta\phi = -\frac{d-2}{2}\omega\phi$. We also need $\delta g^{\mu\nu} = -2\omega g^{\mu\nu}$ and $\delta\sqrt{-g} = d\omega\sqrt{-g}$. Figuring out how the Ricci scalar shifts is a bit of a pain. One nice computer package for doing so is called xAct. The result is that

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\,2\omega\left(\frac{d-2}{2}R - (d-1)\Box\right) \ .$$

We find then that

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^d x \Big(2(\partial^\mu\phi)(\partial_\mu\delta\phi)\sqrt{-g} + (\partial_\mu\phi)(\partial_\nu\phi)\delta(\sqrt{-g}g^{\mu\nu}) + \\
&\qquad\qquad + 2\xi R\phi\delta\phi\sqrt{-g} + \xi\phi^2\delta(\sqrt{-g}R)\Big) \\
&= \int d^d x\sqrt{-g}\Big[\frac{d-2}{2}(\partial^\mu\phi)\partial_\mu(\omega\phi) - \frac{d-2}{2}\omega(\partial^\mu\phi)(\partial_\mu\phi) + \\
&\qquad\qquad + \frac{d-2}{2}\xi R\omega\phi^2 - \frac{d-2}{2}\xi R\omega\phi^2 + (d-1)\xi\omega\Box\phi^2\Big] \ .
\end{aligned}$$

Integrating the first term by parts, we see that the first and second terms cancel against the last term, provided $\xi = \frac{d-2}{4(d-1)}$. The third and fourth terms cancel trivially.

The following two questions assume a normalization of the boundary anomaly coefficients that follows from the following anomalous scale variation of the effective action

$$\delta_\sigma \mathcal{W} = \int d^2 x \sqrt{\det h}\left(a R_{(2d)} + b\hat{K}_{\mu\nu}\hat{K}^{\mu\nu}\right)\sigma$$

where $R_{(2d)}$ is the Ricci scalar curvature on the boundary and $\hat{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{2}h_{\mu\nu}K$ is the traceless part of the extrinsic curvature.

5. ***Boundary "a" for the Free Scalar:** Compute the boundary coefficient a for a conformally coupled three dimensional scalar field by computing the partition function on a hemisphere with both Dirichlet and Neumann boundary conditions.

The eigenmodes on an S^3 are characterized by three integers (ℓ, m, m') subject to $\ell \geq |m| \geq |m'|$. The eigenvalues are $-\frac{\ell(\ell+2)}{R_0^2}$. The degeneracy is then

$$\deg = \sum_{m=0}^{\ell} \sum_{m'=-m}^{m} 1 = \sum_{m=0}^{\ell} (2m+1) = (\ell+1)^2$$

The eigenmodes on the sphere are also eigenmodes on the hemisphere. Those with $\ell - m \in 2\mathbb{Z}+1$ are odd functions about the equator and hence satisfy Dirichlet boundary conditions. Those with $\ell - m \in 2\mathbb{Z}$ are even and thus are Neumann. The Dirichlet and Neumann degeneracies are respectively

$$\begin{aligned} \deg_D &= \frac{1}{2}\ell(\ell+1) , \\ \deg_N &= \frac{1}{2}(\ell+1)(\ell+2) . \end{aligned}$$

Reassuringly, they add up to $(\ell+1)^2$. To derive these numbers, one can specialize to even and odd ℓ.

Now we have a conformally coupled scalar. In an earlier problem, we saw that the conformal coupling constant $\xi = \frac{d-2}{4(d-1)} = \frac{1}{8}$ in the three dimensional case. Also, a unit S^3 has Ricci scalar curvature equal to 6. The conformal coupling thus acts like a mass which will shift the eigenvalue by $\frac{3}{4}$, giving

$$\ell(\ell+2) + \frac{3}{4} = \left(\ell + \frac{1}{2}\right)\left(\ell + \frac{3}{2}\right) .$$

At any rate, we need to evaluate the infinite (and divergent!) sums

$$\begin{aligned} W_D &= \frac{1}{2}\sum_{\ell=0}^{\infty} \frac{1}{2}\ell(\ell+1)\log\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right) , \\ W_N &= \frac{1}{2}\sum_{\ell=0}^{\infty} \frac{1}{2}(\ell+1)(\ell+2)\log\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right) . \end{aligned}$$

We can also consider the S^3 partition function while we are at it,

$$W = \frac{1}{2}\sum_{\ell=0}^{\infty} (\ell+1)^2 \log\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right) .$$

A trick that we have at our disposal now, since we are interested in boundary anomalies, is to regulate by removing the S^3 partition function:

$$W_D - \frac{1}{2}W = -\frac{1}{4}\sum_{\ell=0}^{\infty} (\ell+1)\log\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right) .$$

As $W_D + W_N = W$, we will have $W_D - \frac{1}{2}W = -W_N + \frac{1}{2}W$.

As in the S^2 case, we will use a heat kernel inspired regularization

$$W_D - \frac{1}{2}W = -\frac{1}{4}\sum_{\ell=0}^{\infty}(\ell+1)\int \frac{dt}{t}(e^{-(\ell+\frac{1}{2})t} + e^{-(\ell+\frac{3}{2})t}) = -\frac{1}{4}\int \frac{dt}{t}\frac{e^{t/2}(1+e^t)}{(1-e^t)^2} \ .$$

We again anticipate the anomaly comes from the small t (or UV) region of the integral –

$$W_D - \frac{1}{2}W \sim -\frac{1}{4t_{\rm min}^2} + \frac{1}{48}\log t_{\rm min} \ .$$

Again, we take $t_{\rm min} = 1/R_0\Lambda_{UV}$. We can remove the leading $R_0^2\Lambda_{UV}^2$ behavior through a boundary volume counter term. The remaining $\Lambda_{UV}\partial_{\Lambda_{UV}}W = -\frac{1}{48}$ we interpret as being proportional to the boundary anomaly a coefficient. (The Neumann case will have the opposite sign.) In our normalization $8\pi a = -\frac{1}{48}$. We may have made a number of sign mistakes along the way, but I know the final answer is correct since the Dirichlet case should have a lower a than the Neumann one. a is monotonically decreasing under boundary RG flows.

6. ***Boundary "b" for the Free Scalar:** Compute the boundary coefficient b for the conformally coupled three dimensional scalar by computing the two point function of the displacement operator, both in the case of Neumann and Dirichlet boundary conditions.

Here we need to compute the stress tensor for the conformally coupled scalar and then use the boundary limit of the T^{nn} component as our displacement operator. In a general curved space-time, the stress tensor has the form

$$T^{\mu\nu} = 2\sqrt{-g}\frac{\delta S}{\delta g_{\mu\nu}} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}((\partial_\rho\phi)(\partial^\rho\phi) + \xi R\phi^2) + \xi(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\phi^2 - \xi(\nabla^\mu\nabla^\nu - \eta^{\mu\nu}\Box)\phi^2 \ .$$

In flat space, this result reduces to

$$T^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi) - \xi(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)\phi^2 \ .$$

The last term is invisible from the standard derivation, using Noether's theorem and translation invariance. It is however a total derivative and corresponds to a standard "improvement" term.

For Neumann boundary conditions, the displacement operator takes the form (using the equation of motion)

$$D = -\frac{1}{2}(\partial_a\phi)(\partial^a\phi) + \xi\partial_a\partial^a\phi^2 \ ,$$

while for Dirichlet, we have instead

$$D = \frac{1}{2}(\partial^n\phi)(\partial_n\phi)$$

The displacement operator two-point function then follows from Wick's Theorem and the two-point function for ϕ. By the method of images, we find

$$\langle \phi(x,y)\phi(0,y')\rangle = \kappa\left(\frac{1}{(x^2+(y-y')^2)^{(d-2)/2}} \pm \frac{1}{(x^2+(y+y')^2)^{(d-2)/2}}\right)$$

Here x is the tangential distance along the boundary while y and y' are the perpendicular distance from it. The plus sign is for Neumann and the minus for Dirichlet. Finally, a conventional normalization is $\kappa^{-1} = (d-2)\mathrm{Vol}(S^{d-1})$, consistent with the 1/2 in front of the action, where $\mathrm{Vol}(S^d)$ is the volume of a unit S^d sphere.

Two useful auxiliary results are

$$\begin{aligned}
\langle \partial_a \phi(x,0)\partial^a \phi(x',0)\rangle &= -\frac{2\kappa(d-2)}{(x-x')^d} \\
\langle \partial_n \phi(x,0)\partial^n \phi(x',0)\rangle &= \frac{2\kappa(d-2)}{(x-x')^d}
\end{aligned}$$

From these, we obtain that the displacement operator two-point functions,

$$\langle D(x)D(0)\rangle_{\text{Dirichlet}} = \langle D(x)D(0)\rangle_{\text{Neumann}} = \frac{2\kappa^2(d-2)^2}{(x-x')^{2d}} \ ,$$

are the same.