Supergravity a la Fin de Siècle

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1 Introduction

One of my most romantic images is that of turn of the Century Europe. That is the period in the late 1800's into the early 1900's. Scientifically the great leap forward in Fundamental Physics that we now all know about had not yet happened, although various seeds had already begun sprouting. On the other hand the art scene was changing rapidly and there was great excitement and a sense of anticipation of nascent change. Or at least I like to believe that was the case.

These lectures are on Supergravity. Supergravity as understood at another turn of the century one hundred years later. The art scene may not prove to be as revolutionary; it was dominated by Brit Pop. But the science may well turn out to be. Hopefully you will know by the next fin de sciècle.

In these lectures I aim to introduce supergravity as it was understood then. The scientific revolution on the doorstep is of course the AdS/CFT correspondence. In the interests of time we will not discuss that. It requires its own lecture series by a person more qualified than me.

Supergravity holds a split personality. On the one had it is merely a low energy effective action of String Theory whereas on the other it contains exact information in the String coupling constant.[1] Thus although Supergravity was once the poorer sibling to the UV complete perturbative expansion offered by String Theory it also offers the best window we have into the non-perturbative nature of String Theory and ultimately M-Theory. A notable early example of this was the discovery of mirror symmetry of Calabi-Yau manifolds, something that was unexpected, deep and born from Supergravity. In short it is an effective theory of Strings, valid below the string scale. And, as with other areas of Physics, effective field theories are very insightful and powerful.

So what will we discuss? In section 2 we will introduce the basics of Supergravity: what is it exactly? An important point here is how to describe fermions (spinors) on a curved spacetime through the vielbein and spin connection formalism. In section 3 we will look at examples of supergravities with maximal supersymmetry in ten and eleven dimensions, as is relevant for String Theory and M-theory. We will look at special, so-called BPS, solutions which preserve some fraction of the supersymmetry. These are typically known as p-branes and which include the famous example of Dp-branes of String Theory. As an aside we will prove the non-perturbative stability of these states using the Nestor tensor in a method adapted from Witten's proof of the stability of Minkowski space. Lastly in section four we will look at another aspect of higher dimensional supergravities: namely their compactification on torii and the appearance of U-duality.

Topics that we won't discuss, mainly due to time and my ignorance, are gauged

[1] Thats isn't really fair as there are import phenomenological applications of four-dimensional $N = 1$ supergravity, independently of whether or not String Theory has anything to do with our Universe, that we won't discuss here.

supergravities, G-structures, classification of supersymmetric solutions and generalised geometry. A more complete discussion can be found in a variety of text books but perhaps most notably [1] and [2]. I also mention [3] which is an old introduction to M-theory.

2 Vielbeins and Spin Connections

First let us describe what we mean by supergravity. One way to think about it is a theory with local supersymmetry. What has this got to do with gravity? Well consider a simple supersymmetric system of the form (for simplicity we assume everything is real)

$$\begin{aligned} \delta\phi &= i\bar{\epsilon}\psi \\ \delta\psi &= i\Gamma^\mu\partial_\mu\phi + \dots \ , \end{aligned} \tag{1}$$

where $\bar{\epsilon} = \epsilon^T C^{-1}$ with C the charge conjugation matrix:

$$\Gamma_\mu^T = -C\Gamma_\mu C^{-1} \tag{2}$$

and the ellipsis denotes additional terms which might arise in an interacting theory. We don't particularly care what ϕ and ψ represent (scalars, components of vectors or spinors *etc.*) only that ϕ is Bosonic and ψ Fermionic. Looking at the closure of two supersymmetries on ϕ we find

$$\begin{aligned} [\delta_1, \delta_2]\phi &= i\bar{\epsilon}_2\Gamma^\mu\epsilon_1\partial_\mu\phi - (1 \leftrightarrow 2) + \dots \\ &= v^\mu\partial_\mu\phi + \dots \ , \end{aligned} \tag{3}$$

where $v^\mu = i\bar{\epsilon}_2\Gamma^\mu\epsilon_2 - i\bar{\epsilon}_1\Gamma^\mu\epsilon_1$ is some constant vector. We can think of this resulting transformation as arising from a translation:

$$\begin{aligned} x^\mu &\to x^\mu + v^\mu \\ \phi(x^\mu) &\to \phi(x^\mu + v^\mu) \ . \end{aligned} \tag{4}$$

This is the familiar statement that (Poincarè) supersymmetries close onto momentum:

$$\{Q_\alpha, Q_\beta\} = 2(\Gamma^\mu C^{-1})_{\alpha\beta}P_\mu \tag{5}$$

If we now consider a local supersymmetry where ϵ depends on spacetime our translation becomes an infinitesimal coordinate transformation

$$x^\mu \to x^\mu + v^\mu(x) \ . \tag{6}$$

Thus theories that are invariant under local supersymmetry must also posses general coordinate invariance.

What might such a theory look like. We expect it to have a metric tensor $g_{\mu\nu}$ how would that change under supersymmetry? A natural guess would be

$$\delta g_{\mu\nu} \propto i\bar{\epsilon}\Gamma_\mu\psi_\nu + i\bar{\epsilon}\Gamma_\nu\psi_\mu \tag{7}$$

where now ψ_μ, the superpartner to $g_{\mu\nu}$, has picked-up a vector index in addition to its spinor index α which we have not written explicitly. Such a field is called a gravitino or Raita-Schwinger field. What would the variation of ψ_μ be? We would need something like

$$\delta\psi_\mu = D_\mu\epsilon + \dots \tag{8}$$

But what does D_μ mean exactly? How do we put spinors on a curved manifold with metric $g_{\mu\nu}$. This leads us to construct vielbeins and spin connections.

2.1 Vielbeins

We know how to define spinors on flat space $\mathbb{R}^D$. Under an infinitesimal Lorentz transformation $x^\mu \to x^\mu + \lambda^\mu{}_\nu x^\nu$ vectors transform as $\delta V^\mu = \lambda^\mu{}_\nu V^\nu$ and spinors as

$$\delta\psi = \frac{1}{4}\lambda^{\mu\nu}\Gamma_{\mu\nu}\psi \tag{9}$$

We need to find a way to map this structure onto a manifold.

Exercise: Verify that if $A_\mu = \bar{\psi}\Gamma_\mu\psi$ then $\delta A_\mu = \lambda_\mu{}^\nu A_\nu$.

On a manifold the tangent space at each point is $\mathbb{R}^D$. So we can define spinors there. We then somehow need to move them down to the manifold itself. To do this we introduce the concept of a vielbein frame. Here we write

$$g_{\mu\nu} = e_\mu{}^{\underline{\rho}} e_\nu{}^{\underline{\lambda}} \eta_{\underline{\rho}\underline{\lambda}} \tag{10}$$

For some object $e_\mu{}^{\underline{\nu}}$ which is called the vielbein (or sometimes vierbein in four-dimensions, zweibein in two-dimensions and einbein in one-dimension). Since $g_{\mu\nu}$ is invertible so is $e_\mu{}^{\underline{\rho}}$ and we denote its inverse by $e_{\underline{\rho}}{}^\mu$ so that

$$e_\mu{}^{\underline{\rho}} e_{\underline{\rho}}{}^\nu = \delta_\mu^\nu \tag{11}$$

or equivalently

$$e_{\underline{\lambda}}{}^\mu e_\mu{}^{\underline{\rho}} = \delta_{\underline{\lambda}}^{\underline{\rho}} \tag{12}$$

The main point of vielbeins is that they allow us to convert tangent space indices, which we denote with an underline $\underline{\mu}$, into so-called world-indices corresponding to coordinates x^μ on the manifold. Since we can define spinors in the tangent space we can then map them to the manifold. We can use $e_\mu{}^{\underline{\rho}}$ to map between objects on the manifold such as vectors and their cousins in the tangent space:

$$V^{\underline{\rho}} = e_\mu{}^{\underline{\rho}} V^\mu \tag{13}$$

Note the if there are functions $y^{\underline{\mu}}$ such that

$$e_\mu{}^{\underline{\nu}} = \frac{\partial y^{\underline{\nu}}}{\partial x^\mu} \tag{14}$$

then actually there is a coordinate transformation so that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\underline{\mu\nu}} dy^\mu dy^\nu \tag{15}$$

and therefore we have flat space. Thus (14) is typically not possible. As such if we write

$$\partial_{\underline{\nu}} = e_{\underline{\nu}}{}^\mu \partial_\mu \tag{16}$$

then $\partial_{\underline{\nu}}$ is not a derivative with respect to some variable, in particular $[\partial_{\underline{\mu}}, \partial_{\underline{\nu}}] \neq 0$. The $e_\mu{}^{\underline{\nu}}$ are sometimes referred to as a non-coordinate frame.

In the literature it is more common to denote a vielbein by $e_\mu{}^a$. While I agree that this is a more elegant notation I prefer not to use it as i) it requires taking up a new index a whose range is exactly the same as μ and ii) if you ever need write down an explicit expression for, say V^1, without the underline it is hard to know whether you refer to the tangent or world frame.

Given a metric $g_{\mu\nu}$ one can always find a vielbein. Since there is no symmetry condition on $e_\mu{}^{\underline{\nu}}$ it holds more degrees of freedom than in $g_{\mu\nu}$. On the other hand it is not unique as

$$e'_\mu{}^{\underline{\nu}} = \Lambda^{\underline{\nu}}{}_{\underline{\rho}}(x) e_\mu{}^{\underline{\rho}} \tag{17}$$

will also satisfy

$$g_{\mu\nu} = e'_\mu{}^{\underline{\rho}} e'_\nu{}^{\underline{\lambda}} \eta_{\underline{\rho\lambda}} \tag{18}$$

for any local Lorentz transformation $\Lambda^{\underline{\nu}}{}_{\underline{\rho}}(x)$:

$$\eta_{\underline{\rho\lambda}} = \Lambda^\nu{}_{\underline{\rho}}(x) \Lambda^\nu{}_{\underline{\lambda}}(x) \eta_{\underline{\mu\nu}} \tag{19}$$

One can check that the counting is right: In D-dimensions $e_\mu{}^{\underline{\nu}}$ has D^2 independent components but the redundancy corresponding to local Lorentz transformations relates $D(D-1)/2$ degrees of freedom leaving $D(D+1)/2$ degrees of freedom which matches that of a symmetric $D \times D$ matrix.

As an aside it is natural to think of $e_\mu{}^{\underline{\nu}}$ as defining a 1-form that takes values in the tangent space:

$$e^{\underline{\nu}} = e_\mu{}^{\underline{\nu}} dx^\mu \tag{20}$$

If you know forms this makes life a little simpler. If you don't it's okay, your life will just be more complicated until you do.

Next we need to introduce the notion of a covariant derivative otherwise known as a connection. The Levi-Civita connection is uniquely determined by the conditions

$$\begin{aligned} D_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} g_{\mu\rho} = 0 \\ \Gamma^\rho_{\mu\nu} &= \Gamma^\rho_{\nu\mu} \end{aligned} \tag{21}$$

What happens here? Well we want to impose the analogous condition

$$D_\mu e_\nu{}^{\underline{\lambda}} = \partial_\mu e_\nu{}^{\underline{\rho}} - \Gamma^\rho_{\mu\nu} e_\rho{}^{\underline{\lambda}} + \omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}} e_\nu{}^{\underline{\rho}} = 0 \tag{22}$$

where

$$\omega_\mu{}^{\underline{\sigma\tau}} = -\omega_\mu{}^{\underline{\tau\sigma}} \tag{23}$$

takes values in the Lie algebra of the local Lorentz group.

Here we have done the usual physicist's trick of introducing a connection for each type of index (with a plus or minus sign depending on whether or not it appears upstairs or downstairs respectively). We still want $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ so we have

$$D_\mu e_\nu{}^{\underline{\lambda}} - D_\nu e_\mu{}^{\underline{\lambda}} = \partial_\mu e_\nu{}^{\underline{\rho}} - \partial_\nu e_\mu{}^{\underline{\rho}} + \omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}} e_\nu{}^{\underline{\rho}} - \omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}} e_\nu{}^{\underline{\rho}} = 0 \tag{24}$$

If you know forms then this is just

$$de^{\underline{\lambda}} + \omega^{\underline{\lambda}}{}_{\underline{\rho}} \wedge e^{\underline{\rho}} = 0 \tag{25}$$

where $\omega^{\underline{\lambda}}{}_{\underline{\rho}} = \omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}} dx^\mu$. This is enough to determine $\omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}}$ given $e_\nu{}^{\underline{\rho}}$. In fact there is a formula for $\omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}}$ in terms of $e_\nu{}^{\underline{\lambda}}$:

$$\omega_\mu{}^{\underline{\lambda\rho}} = 2e^{\nu[\underline{\lambda}}\partial_{[\mu} e_{\nu]}{}^{\underline{\rho}]} - e^{\nu[\underline{\lambda}} e^{\underline{\rho}]\sigma} e_{\mu\underline{\tau}} \partial_\nu e_\sigma{}^{\underline{\tau}} \tag{26}$$

However I don't usually find it much use in practice, I just solve (25) in a case by case basis. More generally the right hand side of (25) is given by the torsion tensor and, in supergravity, this is typically non-zero in the presence of Fermionic fields

It is also easy to see that under a local Lorentz transformation $e'_\mu{}^{\underline{\nu}} = \Lambda^{\underline{\nu}}{}_{\underline{\rho}}(x) e_\mu{}^{\underline{\rho}}$ we have

$$\omega'_\mu{}^{\underline{\nu}}{}_{\underline{\rho}} = \Lambda^{\underline{\nu}}{}_{\underline{\sigma}} \partial_\mu (\Lambda^{-1})^{\underline{\sigma}}{}_{\underline{\rho}} + \Lambda^{\underline{\nu}}{}_{\underline{\sigma}} \omega_\mu{}^{\underline{\sigma}}{}_{\underline{\tau}} (\Lambda^{-1})^{\underline{\tau}}{}_{\underline{\rho}} \tag{27}$$

so, as expected, it transforms like a connection. In particular it transforms like an gauge theory connection for the Lie algebra of $SO(1, D-1)$.

Knowing $\omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}}$ we can also deduce $\Gamma^\lambda_{\mu\nu}$ from (22) and it is indeed given by the usual formula:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \ . \tag{28}$$

Exercise: Show this (Hint: think about $D_\lambda g_{\mu\nu} = 0$ in terms of vielbeins first before rushing into a painful calculation).

2.2 Aside: Differential forms

It is helpful to introduce the notion of a differential r-form which in components is simply a $(0, r)$ tensor that is totally anti-symmetric:

$$\omega_{\mu_1 \ldots \mu_r} = \omega_{[\mu_1 \ldots \mu_r]} \tag{29}$$

The form is written as

$$\omega = \frac{1}{r!}\omega_{\mu_1...\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r} \tag{30}$$

where dx^μ is a basis for the cotangent space and $\wedge$ is a totally anti-symmetric tensor product:

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r} = dx^{\mu_1} \otimes dx^{\mu_2} \otimes ... \otimes dx^{\mu_r} \pm cyclic \tag{31}$$

This allows for the notion of a wedge product of an r-form and s-form to give an $(r+s)$-form:

$$\begin{aligned}\omega \wedge \rho &= \frac{1}{r!s!}\omega_{\mu_1...\mu_r}\rho_{\nu_1...\nu_s}(dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r}) \wedge (dx^{\nu_1} \wedge dx^{\nu_2} \wedge ... \wedge dx^{\nu_s}) \\ &= \frac{1}{(r+s)!}(\omega \wedge \rho)_{\mu_1...\mu_{r+s}} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{r+s}}\end{aligned} \tag{32}$$

so

$$(\omega \wedge \rho)_{\mu_1...\mu_{r+s}} = \frac{(r+s)!}{r!s!}\omega_{[\mu_1...\mu_r}\rho_{\mu_{r+1}...\nu_{r+s}]} \tag{33}$$

where now the cyclic permutations are over all $r+s$ indices. Note that one finds

$$\omega \wedge \rho = (-1)^{rs}\rho \wedge \omega \tag{34}$$

i.e. even forms commute with each other and odd-forms but two odd-forms will anti-commute.

A key point about forms is that we can define a derivative, the so-called exterior derivative, without the need for a connection:

$$d\omega = \frac{1}{r!}\partial_\nu \omega_{\mu_1...\mu_r} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r} \tag{35}$$

which is a totally anti-symmetric $(0, r+1)$ tensor, *i.e.* an $(r+1)$-form. It is easy to see that differential forms ω are invariant under diffeomorphisms with usual transformation law for their components $\omega_{\mu_1..\mu_r}$ inherited from tensors. And so is $d\omega$, the point is that the totally anti-symmetry kills off the inhomgeneous terms that one finds when transforming $\partial_\nu\omega_{\mu_1...}$. Indeed you could, if it makes you happier, define

$$d\omega = \frac{1}{r!}D_\nu \omega_{\mu_1...\mu_r} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r} \tag{36}$$

where $D_\mu\omega_{\mu_1...\mu_r} = \partial_\nu\omega_{\mu_1...\mu_r} - \Gamma^\lambda_{\nu\mu_1}\omega_{\lambda...\mu_r} + ...$ but the Christoffel connection terms will vanish under anti-symmetry. The most important property of the exterior derivative is that

$$d^2\omega = 0 \tag{37}$$

for any form ω. This easily follows from the fact that $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$.

There is one last operation one can do on forms if one has a metric. This is known as the Hodge dual

$$\star\omega_{\nu_1\ldots\nu_{D-r}} = \frac{1}{r!}\sqrt{|\det g|}\varepsilon_{\nu_1\ldots\nu_{D-r}\mu_{D-r+1}\ldots\mu_D}g^{\mu_1\lambda_1}\ldots g^{\mu_r\lambda_r}\omega_{\lambda_1\ldots\lambda_r} \tag{38}$$

here $\varepsilon_{\nu_1\ldots\mu_D}$ is the totally anti-symmetric tensor with $\varepsilon_{012(D-1)} = -1$ (in Minkowski space). Thus $\star\omega$ is a $D-r$ form.

Lastly we can integrate a D-form over a D-dimensional manifold via the identification

$$\int\omega = \frac{1}{D!}\int\omega_{\mu_1\ldots\mu_D}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_D} \equiv \int\omega_{012\ldots D}dx^0dx^1\ldots dx^{D-1} \tag{39}$$

Using the Hodge-dual we then have a natural inner-product on r-forms:

$$\langle\omega,\rho\rangle = \int\star\omega\wedge\rho \tag{40}$$

exercise: Convince yourself that

$$\int\star dA\wedge dA = -\frac{1}{2}\int d^Dx\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{41}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

2.3 Spin Connections

Next we return to Spinors. First we start with Γ-matrices. In tangent space they satisfy

$$\{\Gamma_{\underline{\mu}},\Gamma_{\underline{\nu}}\} = 2\eta_{\underline{\mu\nu}} \tag{42}$$

and we can choose your favourite representation in terms of constant matrices. All we insist on is that

$$\Gamma^\dagger_{\underline{\mu}} = -\Gamma_{\underline{0}}\Gamma_{\underline{\mu}}\Gamma^{-1}_{\underline{0}} \tag{43}$$

There is also always a matrix C such that

$$\Gamma^T_{\underline{\mu}} = -C\Gamma_{\underline{\mu}}C^{-1} \tag{44}$$

We can now construct

$$\Gamma_\mu = e_\mu{}^{\underline{\nu}}\Gamma_{\underline{\nu}} \tag{45}$$

which will satisfy

$$\{\Gamma_\mu,\Gamma_\nu\} = 2g_{\mu\nu} \tag{46}$$

Lastly we define

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{4}\omega_\mu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\psi \tag{47}$$

.

Exercise: Show that if $A_\mu = \bar{\psi}\Gamma_\mu\psi$ then

$$D_\mu\bar{\psi}\Gamma_\nu\psi + \bar{\psi}\Gamma_\mu D_\nu\psi = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu}A_\sigma$$

This gives us a new way to define a curvature tensor. Let us consider the commutator

$$\begin{aligned}
[D_\mu, D_\nu]\psi &= D_\mu(\partial_\nu\psi + \frac{1}{4}\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\psi) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu\psi + \frac{1}{4}\partial_\mu\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\psi + \frac{1}{4}\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\partial_\mu\psi - \Gamma^\tau_{\mu\nu}D_\tau\psi \\
&+ \frac{1}{4}\omega_\mu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\partial_\nu\psi + \frac{1}{16}\omega_\mu{}^{\underline{\tau\pi}}\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\tau\pi}}\Gamma_{\underline{\lambda\rho}}\psi - (\mu \leftrightarrow \nu) \\
&= \frac{1}{4}\partial_\mu\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\psi + \frac{1}{16}\omega_\mu{}^{\underline{\tau\pi}}\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\tau\pi}}\Gamma_{\underline{\lambda\rho}}\psi - (\mu \leftrightarrow \nu)
\end{aligned} \tag{48}$$

since the terms involving derivatives of ψ cancel. Next we use the identity

$$\Gamma_{\underline{\tau\pi}}\Gamma_{\underline{\lambda\rho}} = \Gamma_{\underline{\tau\pi\lambda\rho}} + 2\cdot 2\eta_{\underline{\pi\lambda}}\Gamma_{\underline{\tau\rho}} + 2\eta_{\underline{\pi\lambda}}\eta_{\underline{\tau\rho}} \tag{49}$$

where it is understood that the indices are anti-symmeterised in $\underline{\pi\lambda}$ and $\underline{\tau\rho}$. The first and last terms cancel out under $\mu \leftrightarrow \nu$ so

$$\frac{1}{16}\omega_\mu{}^{\underline{\tau\pi}}\omega_\nu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\tau\pi}}\Gamma_{\underline{\lambda\rho}}\psi - (\mu \leftrightarrow \nu) = \frac{1}{4}\omega_\mu{}^{\underline{\tau\lambda}}\omega_{\nu\underline{\lambda}}{}^{\underline{\rho}}\Gamma_{\underline{\tau\rho}}\psi - (\mu \leftrightarrow \nu) \tag{50}$$

Thus we have

$$[D_\mu, D_\nu]\psi = \frac{1}{4}\Gamma^{\underline{\lambda\rho}}R_{\mu\nu\underline{\lambda\rho}}\psi \tag{51}$$

with

$$R_{\mu\nu}{}^{\underline{\lambda}}{}_{\underline{\rho}} = \partial_\mu\omega_\nu{}^{\underline{\lambda}}{}_{\underline{\rho}} - \partial_\nu\omega_\mu{}^{\underline{\lambda}}{}_{\underline{\rho}} + \omega_\mu{}^{\underline{\lambda}}{}_{\underline{\sigma}}\omega_\nu{}^{\underline{\sigma}}{}_{\underline{\rho}} - \omega_\nu{}^{\underline{\lambda}}{}_{\underline{\sigma}}\omega_\mu{}^{\underline{\sigma}}{}_{\underline{\rho}} \tag{52}$$

In terms of forms we have

$$R^{\underline{\lambda}}{}_{\underline{\rho}} = d\omega^{\underline{\lambda}}{}_{\underline{\rho}} + \omega^{\underline{\lambda}}{}_{\underline{\sigma}} \wedge \omega^{\underline{\sigma}}{}_{\underline{\rho}} \tag{53}$$

It turns out that

$$R_{\mu\nu\lambda}{}^{\rho} = e_{\lambda\underline{\sigma}}e^{\rho\underline{\tau}}R_{\mu\nu}{}^{\underline{\sigma}}{}_{\underline{\tau}} \tag{54}$$

is the usual Riemann curvature tensor, where we raise and lower underlined indices using η.

Exercise: Show this (Hint: think about $[D_\mu, D_\nu]A_\lambda$ where $A_\lambda = i\bar{\epsilon}\Gamma_\lambda\psi$).

This is nice as the relation to gauge theory is closer: The Riemann tensor is a two-form that takes values in the Lie-algebra of $SO(1, D-1)$, the structure group of the tangent space (as opposed to the field strength in a gauge theory which is a two-form that takes values in the Lie-algebra of the gauge group).

2.4 Elementary Supergravity

So we are ready to write down an attempt at a supergravity. Our guess is

$$S_{guess} = \frac{1}{\kappa^2}\int d^D x\sqrt{-g}(\frac{1}{2}R + \frac{i}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\lambda}D_\nu\psi_\lambda) \tag{55}$$

The second term is known as a Raita-Schwinger term and we can take

$$D_\mu\psi_\nu = \partial_\mu\psi_\nu + \frac{1}{4}\omega_\mu{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\psi_\nu \tag{56}$$

Note the absence of a Christoffel symbol for the μ index. Here D_μ acting on spinors is a local Lorentz covariant derivative. Even if we had included the Christoffel symbol term in $D_\nu\psi_\lambda$ then it would drop out from the action since it is symmetric in $\nu\lambda$, as is the case with differential forms; indeed $\psi_\mu dx^\mu$ can be thought of as a spinor-valued 1-form.

Does it admit a supersymmetry of the form we discussed above? For concreteness let us assume that ψ_μ and Γ_μ are real with $\bar{\psi} = \psi^T C$ and $C = \Gamma_{\underline{0}}$. This is valid in four, ten and eleven dimensions, which we will be most interested in, but not in general. Let us look at the variation

$$\delta S_{guess} = \frac{1}{\kappa^2}\int d^D x\sqrt{-g}\Big[\frac{1}{2}\delta g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \frac{i}{2}\delta\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\psi_\lambda + \frac{i}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda\Big] + \ldots \tag{57}$$

Here the ellipsis are terms that come from varying metric and connection terms in the Raita-Schwinger term. Since the variation of a Boson is a Fermion these terms will be cubic in Fermions. Thus we can look at the terms we have which are all linear in Fermions and hence must vanish independently of the cubic ones.

First we use some Γ-matrix manipulations to combine the last two terms:

$$\begin{aligned}\delta\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\psi_\lambda &= D_\nu\bar{\psi}_\lambda\Gamma^{\mu\nu\lambda}\delta\psi_\mu \\ &= -\psi_\lambda\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\mu + D_\nu(\bar{\psi}_\lambda\Gamma^{\mu\nu\lambda}\delta\psi_\mu) \\ &= \psi_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda + D_\nu(\bar{\psi}_\lambda\Gamma^{\mu\nu\lambda}\delta\psi_\mu)\end{aligned} \tag{58}$$

The first line follows from the fact that $C\Gamma^{\mu\nu\lambda}$ is an anti-symmetric matrix but Fermions anti-commute. Thus, dropping a boundary term, we have

$$\delta S_{guess} = \frac{1}{\kappa^2}\int d^D x\sqrt{-g}\Big[\frac{1}{2}\delta g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + i\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda\Big] + \ldots \tag{59}$$

Next we put in our guess $\delta\psi_\lambda = D_\lambda\epsilon$ and do some more manipulations:

$$\begin{aligned}i\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda &= i\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu D_\lambda\epsilon \\ &= \frac{i}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}[D_\nu, D_\lambda]\epsilon \\ &= \frac{i}{8}\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}R_{\nu\lambda\underline{\rho\sigma}}\Gamma^{\underline{\rho\sigma}}\epsilon \\ &= \frac{i}{8}\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma}R_{\nu\lambda\rho\sigma}\epsilon\end{aligned} \tag{60}$$

Next we note that

$$\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma} = \Gamma^{\mu\nu\lambda\rho\sigma} + 3\cdot 2\Gamma^{\mu\nu\sigma}g^{\lambda\rho} + 3\cdot 2\cdot 2\Gamma^{\mu}g^{\lambda\rho}g^{\sigma\nu} \tag{61}$$

where it is understood that terms are anti-symmeterised in $\mu\nu\lambda$ and $\rho\sigma$. The first two terms give zero as $R_{[\nu\lambda\rho]\sigma} = 0$ and $R_{\nu\sigma} = R_{\sigma\nu}$. So we are left with

$$i\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda = \frac{12}{8}i\bar{\psi}_\mu\Gamma^{[\mu}g^{\lambda|\rho}g^{\sigma|\nu]}R_{\nu\lambda\rho\sigma}\epsilon \tag{62}$$

where we have put back the anti-symmeterisation (which is not needed for $\rho\sigma$ as $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$). There are only two types of independent terms:

$$\begin{aligned} i\bar{\psi}_\mu\Gamma^{\mu\nu\lambda}D_\nu\delta\psi_\lambda &= \frac{i}{8}\bar{\psi}_\mu\left(2\Gamma^\mu g^{\lambda\rho}g^{\sigma\nu} - 4\Gamma^\lambda g^{\mu\rho}g^{\sigma\nu}\right)R_{\nu\lambda\rho\sigma}\epsilon \\ &= \frac{i}{8}\bar{\psi}_\mu\left(-2\Gamma^\mu R + 4\Gamma^\lambda R_{\lambda\mu}\right)\epsilon \\ &= \frac{i}{2}(R_{\mu\lambda} - \frac{1}{2}g_{\mu\lambda})\Gamma^\lambda\epsilon \end{aligned} \tag{63}$$

Thus we see that we can cancel the variation of the Einstein-Hilbert term by taking

$$\delta g^{\mu\nu} = -\frac{i}{2}\bar{\psi}^\mu\Gamma^\nu\epsilon - \frac{i}{4}\bar{\psi}^\nu\Gamma^\mu\epsilon \tag{64}$$

or equivalently

$$\begin{aligned} \delta g_{\mu\nu} &= -\frac{i}{2}(\bar{\epsilon}\Gamma_\mu\psi_\nu + \bar{\epsilon}\Gamma_\nu\psi_\mu) \\ \delta e_\mu{}^{\underline{\nu}} &= -\frac{i}{2}\bar{\epsilon}\Gamma^{\underline{\nu}}\psi_\mu \end{aligned} \tag{65}$$

This is of course too simple. We have arranged for a symmetry to first order in the Fermions for a supergravity in any dimension, provided that the spinors and Γ-matrices are real (which is sometimes true). This can't be the full story and indeed it isn't. For example there won't be a matching of Bosonic and Fermionic degrees of freedom in general (although there is in four-dimensions). The devil is in the cubic Fermion terms and higher. But I hope its given you a flavour of how it works and all supergravities contain a sector that looks like this. Constructing supergravities in dimensions above three is quite a task, generally involving higher order Fermion terms appearing in the Einstein-Hilbert term through a non-zero torsion as well as additional fields. Even in four dimensions, where no new fields need to be added but additional Fermion terms are needed, was tough (the first proof required an early use of computers in Theoretical Physics) [4, 5]. And in ten and eleven dimensions it was a huge tour-de-force. We won't go into the details here. Happily others constructed these theories in the 1970's and 80's. We will simply take their results with eternal gratitude.

3 Ten and Eleven-Dimensional Supergravities and their BPS solutions: p-branes

3.1 Ten and Eleven-Dimensional Supergravities

We want to focus on String and M-theory which take place in ten and eleven dimensions. It was noted by Nahm [6] that the maximum dimension for supergravity is eleven. At two-derivative order the Lagrangian was constructed by Cremmer, Julia and Scherk [7] using only symmetries. This is in fact a very important point: The maximal supergravity theories are all uniquely determined at two-derivative level simply by the requirement that they admit all of the 32 supersymmetries corresponding the largest spinor dimension compatible with Nahm's classification.

Starting in eleven dimensions one can then compactifiy down on tori to obtain maximally supersymmetric theories in lower dimensions. Indeed the motivation of Cremmer, Julia and Scherk was to construct $N = 8$, $D = 4$ supergravity, which at that time was heralded as the ultimate theory of everything (including by the Maharish Yogi). Its called $N = 8$ supergravity as a minimal spinor in four-dimensions has 4 real components and hence a 32-component real spinor in eleven-dimensions decomposes into 8 four-dimensional ones. it is not possible to add more supersymmetries and maintain a theory with spins not greater than 2.

The great irony of supergravity is that, although the theories are defined by the fact that they admit a supersymmetry between Bosons and Fermions one almost always only winds up working with the purely Bosonic sector of these theories. In the interests of time and conventions we will do the same. This saves and enormous amount of technical details.

3.1.1 Eleven-dimensional Supergravity

The field content consists of

$$g_{\mu\nu}, C_{\mu\nu\lambda}, \psi_\mu$$

where $C_{\mu\nu\lambda}$ is totally anti-symmetric (it is a 3-form) and ψ_μ is a Majorana (real) spinor. As we noted it was constructed in [7] and the Bosonic part of the action takes the form

$$S_{11D} = \frac{1}{\kappa^2}\int\left(\frac{1}{2}R\star 1 - \frac{1}{4}\star dC\wedge dC - \frac{1}{12}C\wedge dC\wedge dC\right) \tag{66}$$

Here the expression $\star 1 = \sqrt{-g}dx^0\wedge ...\wedge dx^{10}$ is the Hodge dual of the constant 0-form and is sometimes called the volume form

3.1.2 Ten-Dimensional Type IIA Supergravity

The field content consists of [9, 8]

$$g_{\mu\nu}, A_\mu, \phi, B_{\mu\nu}, C_{\mu\nu\lambda}, \psi_\mu, \chi$$

where $B_{\mu\nu}$ and $C_{\mu\nu\lambda}$ are totally anti-symmetric, χ and ψ_μ are Majorana spinors with both chiralities (indeed from reduction $\chi = \psi_{10}$). However now $\mu = 0, 1, .., 9$.

This theory simply arises from eleven-dimensional supergravity if we assume that no fields depend on x^{10} and decompose the metric and fields as

$$
\begin{aligned}
ds^2_{11} &= e^{-2\phi/3} g_{\mu\nu} dx^\mu dx^\nu + e^{4\phi/3}(dx^{10} + A_\mu dx^\mu) \\
B_{\mu\nu} &= C_{\mu\nu 10} \\
\chi &= \psi_{10}
\end{aligned}
\tag{67}
$$

The remaining fields $C_{\mu\nu\lambda}$ and ψ_μ are the same as the eleven-dimensional theory but where the range of μ is restricted to be $\mu = 0, ..., 9$. With this reduction the Bosonic part of the action is

$$
\begin{aligned}
S_{IIA} = &\frac{1}{\kappa^2} \int e^{-2\phi} \left(\frac{1}{2} R \star 1 - 2 \star d\phi \wedge d\phi + \frac{1}{4} \star dB \wedge dB \right) \\
&+ \frac{1}{\kappa^2} \int \left(\frac{1}{4} \star dA \wedge dA + \frac{1}{4} \star (dC - A \wedge dB) \wedge (dC - A \wedge dB) - \frac{1}{4} B \wedge dC \wedge dC \right)
\end{aligned}
\tag{68}
$$

3.1.3 Ten-Dimensional Type IIB Supergravity

This theory's construction was also a tour-de-force using sophisticated superspace methods [10, 11, 12]. The field content consists of

$$
g_{\mu\nu}, a, \phi, B^a_{\mu\nu}, C^+_{\mu_1 ... \mu_4}, \psi^{a+}_\mu, \chi^-
$$

where $B^a_{\mu\nu}$ and $C^+_{\mu\nu\lambda\rho}$ are totally anti-symmetric and the field strength of C^+ is constrained to be self-dual: $dC^+ = \star dC^+$. Here $a = 1, 2$ and $\Gamma_{11}\psi^{a+} = \psi^{a+}$ while $\Gamma_{11}\chi^- = -\chi^-$.

Formally there is no action for this theory due to the presence of the self-dual field dC^+. Often one writes down an action and then imposes the constraint $dC^+ = \star dC^+$ by hand.

$$
\begin{aligned}
S_{IIB} = &\frac{1}{\kappa^2} \int e^{-2\phi} \left(\frac{1}{2} R \star 1 - 2 \star d\phi \wedge d\phi + \frac{1}{4} \star dB^1 \wedge dB^1 \right) \\
&+ \frac{1}{\kappa^2} \int \left(\frac{1}{4} \star da \wedge da + \frac{1}{4} \star (dB^2 - a dB^1) \wedge (dB^2 - a dB^1) - \frac{1}{4} C^+_4 \wedge dB^1 \wedge dB^2 \right) \\
&+ \frac{1}{\kappa^2} \int \frac{1}{4} \star (dC^+ - \tfrac{1}{2} B^2 \wedge dB^1 + \tfrac{1}{2} B^1 \wedge dB^2) \wedge (dC^+ - \tfrac{1}{2} B^2 \wedge dB^1 + \tfrac{1}{2} B^1 \wedge dB^2)
\end{aligned}
\tag{69}
$$

The theory posses a novel and important $SL(2, \mathbb{R})$ symmetry that rotates the a index and acts as a fractional linear transformation on

$$
\tau = a + i e^{-\phi} \tag{70}
$$

This is not apparent as written here. Rather we need to go to the Einstein frame:

3.1.4 The Einstein Frame

In these examples the Einstein-Hilbert term comes with a coefficient of $e^{-2\phi}$. This matches what we expect from String perturbation theory but is not how we usually write gravitational theories. It is referred to as the string-frame. To get something more familiar we introduce

$$g_{\mu\nu}^{(s)} = e^{\frac{4}{D-2}\phi} g_{\mu\nu}^{(E)} \tag{71}$$

where the superscripts refer to the String and Einstein frame respectively (so in the above expressions we were working with $g_{\mu\nu}^{(s)}$). One finds that in terms of $g_{\mu\nu}^{(E)}$ the actions will now look like

$$\frac{1}{2\kappa^2}\int e^{-2\phi}R[g^{(s)}]\star_{g^{(s)}} 1 = \frac{1}{2\kappa^2}\int R[g^{(E)}]\star_{g^{(E)}} 1 + \ldots \tag{72}$$

i.e. we recover the usual Einstein-Hilbert term. There will be various other power of e^{ϕ} in the remaining terms and well as corrections involving $d\phi$. The $SL(2,\mathbb{R})$ symmetry that we mentioned above is more apparent in this frame where $g_{\mu\nu}^{(E)}$ and C^+ are invariant whereas (B^1, B^2) transform as

$$\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \tag{73}$$

and

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{74}$$

(so $g_{\mu\nu}^{(s)}$ transforms in a non-trivial way).

3.1.5 Ten-Dimensional Type I Supergravity

The ten-dimensional supergravities that we have considered have maximal supersymmetry: corresponding to two Majorana-Weyl (real and chiral) spinor generators. We can also construct supergravity with just one chirality. This is the minimal spinor in ten-dimensions. The gravitational field content is

$$g_{\mu\nu}, \phi, B_{\mu\nu}, \psi_\mu^+, \chi^-$$

In addition these can be coupled to a Yang-Mills theory with fields

$$\mathcal{A}_\mu, \lambda^-$$

that take values in the adjoint representation of the gauge group. The theory will then be anomalous as it has chiral spinors. There is also a gravitational anomaly. However the famous Green-Schwartz anomaly cancelation mechanism [13] (which shifts dB) but this restricts the gauge groups to $SO(32)$ or $E_8 \times E_8$.

$$S_{IIA} = \frac{1}{\kappa^2}\int e^{-2\phi}\left(\frac{1}{2}R\star 1 - 2\star d\phi\wedge d\phi + \frac{1}{4}\star dB\wedge dB + \alpha'\mathrm{tr}(\star\mathcal{F}\wedge\mathcal{F})\right) \tag{75}$$

3.2 BPS Solutions

Supersymmetric theories admit special cases of solutions which preserve a fraction of the supersymmetry (a typical solution will break all supersymmetries). These play an important role as they turn out to be stable and some of their properties can be trusted to all orders in the coupling constant due to the fact that they saturate the so-called Bogomoln'yi bound which puts a lower bound of the mass of a state in terms of its charges.

We can look for such solutions in supergravity. That is we want to find Bosonic solutions for which there is a solution to

$$\delta\psi_\mu = 0 \tag{76}$$

for some ϵ (and also set any other Fermion variation to zero). Since the Fermions vanish we automatically have that the variation of the Bosons vanishes. Typically solving this equation (and any Bianchi identities) is enough to solve the equations of motion (but not always as we will see with the pp-wave).

3.2.1 Special Holonomy Manifolds

One such family of solutions is simply to turn off all fields except the metric. We are then looking for manifolds and metrics which admit a solution to

$$D_\mu\epsilon = 0 \tag{77}$$

Such a spinor is called a Killing-spinor. The most common application is compactification where we imagine a spacetime of the form

$$\mathbb{R}^{1,d-1} \times \mathcal{M} \tag{78}$$

where $\mathcal{M}$ is thought of compact manifold, so small that we can't observe it with present day accelerators. If we want the low energy effective theory to have some supersymmetry then we need solve $D_\mu\epsilon = 0$ for some spinor of the form

$$\epsilon = \varepsilon \otimes \eta \tag{79}$$

where ε is a spinor on $\mathbb{R}^{1,d-1}$ and η a *commuting* spinor on $\mathcal{M}$. In particular we need

$$D_m\eta = 0 \tag{80}$$

where m labels the coordinates of $\mathcal{M}$. It then follows that

$$\frac{i}{4}R_{mn\underline{pq}}\gamma^{\underline{pq}}\eta = 0 \tag{81}$$

and hence the curvature of the manifold is restricted. In particular $\frac{i}{4}R_{mn\underline{pq}}\gamma^{\underline{pq}}$ generates an infinitesimal element of the holonomy group $H \subset SO(n)$ (that is the set of all possible rotations you generate by parallel transport around a closed loop). But if there is a covariantly constant spinor we see that H cannot be all of $SO(n)$. Such manifolds are known as special holonomy manifolds. There is list of such manifolds and groups due to Berger.

Note that if we contract $R_{mn\underline{pq}}\gamma^{\underline{pq}}\eta = 0$ with γ^n we and use properties of the Riemann tensor we find

$$\begin{aligned} 0 = \gamma^n R_{mn\underline{pq}}\gamma^{\underline{pq}}\eta &= R_{mnpq}\gamma^n\gamma^{pq}\eta \\ &= R_{mnpq}(\gamma^{npq} + g^{np}\gamma^q - g^{nq}\gamma^p)\eta \\ &= -2R_{mp}\gamma^p\eta \end{aligned} \tag{82}$$

Thus unless $\gamma^p\eta$ is somehow degenerate these manifolds are always Ricci-flat. And further contraction with γ^m implies $R = 0$ even if somehow the Ricci tensor doesn't vanish.

Given η we can construct tensors:

$$\begin{aligned} V_m &= \bar{\eta}\gamma_m\eta \\ T_{mn} &= \bar{\eta}\gamma_{mn}\eta \\ \Omega_{mnp} &= \bar{\eta}\gamma_{mnp}\eta \qquad etc. \end{aligned} \tag{83}$$

here γ_m are γ-matrices associated to the manifold $\mathcal{M}$ and $\bar{\eta} = \eta^\dagger c$ is the required Dirac conjugate for spinors on $\mathcal{M}$. If $D_m\eta = 0$ then these are constant tensors on $\mathcal{M}$ and that is very restrictive. For example if V_m is not zero then we have a covariantly constant vector. This is stronger than simply a Killing vector and it is not hard to see that it requires that $\mathcal{M}$ is of the form

$$\mathcal{M} = S^1 \times \mathcal{M}' \tag{84}$$

and the metric has no-dependence on the coordinate associated to S^1. So we have some kind of torus reduction. However in some cases we find $V_m = 0$ (if $\mathcal{M}$ is four-dimensional) and $V_m = T_{mn} = 0$ (if $\mathcal{M}$ is six-dimensional) because $c\gamma_m$ and $c\gamma_{mn}$ are anti-symmetric in a suitable sense. In these cases $\mathcal{M}$ is not of the form $S^1 \times \mathcal{M}'$ but we do find a non-zero but constant two-form or three-form respectively. This leads to so-called Calabi-Yau manifolds. Going to six and seven-dimensional manifolds leads to interesting G_2 and $Spin(7)$ special holonomy manifolds respectively. This is a huge topic that we don't have time to do justice to.

It is not known how to construct metrics for compact special holonomy manifolds (except tori with vanishing curvature). If we don't look for compact special holonomy manifolds then there are two relatively simple examples that can be constructed which preserve half of the supersymmetry and which have wide applications.

3.2.2 Plane waves

The easiest supersymmetric solution to construct is that of a plane wave:

$$ds^2 = 2dx^+dx^- + H(dx^+)^2 + (dx^2)^2 + ... + (dx^{D-1})^2 \tag{85}$$

where H is a function of $x^2, ..., x^{D-1}$ only. Let us construct the Killing spinor satisfying $D_\mu \epsilon = 0$. A choice of vielbein is

$$e^{\underline{-}} = dx^- + \frac{1}{2} H dx^+ \qquad e^{\underline{+}} = dx^+ \qquad e^{\underline{I}} = dx^I \tag{86}$$

where $I = 2, ..., D-1$ so that

$$\begin{aligned} ds^2 &= \eta_{\underline{\rho\sigma}} e_\mu{}^{\underline{\rho}} e_\nu{}^{\underline{\sigma}} dx^\mu dx^\nu \\ &= 2 e_\nu{}^{\underline{-}} e_\nu{}^{\underline{+}} dx^\mu dx^\nu + e_\mu{}^{\underline{I}} e_\nu{}^{\underline{I}} dx^\mu dx^\nu \end{aligned} \tag{87}$$

and hence

$$de^{\underline{-}} = \frac{1}{2} \partial_I H dx^I \wedge dx^+ = \frac{1}{2} \partial_I H e^{\underline{I}} \wedge e^{\underline{+}} \qquad de^{\underline{+}} = 0 \qquad de^{\underline{I}} = 0 \tag{88}$$

From which we learn

$$\omega^{\underline{-}}{}_{\underline{I}} = \frac{1}{2} \partial_I H e^{\underline{+}} \tag{89}$$

Note that since $\omega^{\underline{-I}} = -\omega^{\underline{I-}}$, $\omega^{\underline{I}}{}_{\underline{+}} = \omega^{\underline{I-}}$ will be non-zero but $\omega^{\underline{I}}{}_{\underline{+}} \wedge e^{\underline{+}} = 0$. Thus our Killing spinor equation is just

$$\begin{aligned} D_+ \epsilon &= \partial_+ \epsilon + \frac{1}{4} \partial_I H \Gamma_{\underline{-I}} \epsilon = 0 \\ D_- \epsilon &= \partial_- \epsilon = 0 \\ D_I \epsilon &= \partial_I \epsilon = 0 \end{aligned} \tag{90}$$

We can solve this by simply taking ϵ constant and $\Gamma_{\underline{-}} \epsilon = 0$. This is an example where $\Gamma^p \eta$ is degenerate and so we don't automatically find Ricci flat metrics as $R_{\mu+}$ can be non-zero. In fact Ricci flatness requires

$$\partial_I \partial_I H = 0 \tag{91}$$

3.2.3 Multi-Taub-NUT

There is another interesting BPS solution with no other fields turned-on known as Taub-NUT, or more generally Gibbons-Hawking, whose non-trivial part is four-dimensional and Euclidean. So to solve a D-dimensional supergravity we take the remaining dimensions to be those of Minkowski space:

$$\begin{aligned} ds^2 &= H^{-1}(dx^1 + \omega)^2 + H((dx^2)^2 + ... + (dx^4)^2) - (dx^0)^2 + (dx^5)^2 + ...(dx^{D-5})^2 \\ d\omega &= \star_3 dH \end{aligned} \tag{92}$$

In the last line H is only a function of x^2, x^3, x^4, and ω is a 1-form on the associated $\mathbb{R}^3$ plane and $\star_3$ is the Hodge dual on that $\mathbb{R}^3$. However since $d^2 = 0$ we see that

$$\star_3 d \star_3 dH = 0 \tag{93}$$

which is just the Laplace equation $\partial_I \partial_I H = 0$ where now $I = 2, 3, 4$. I leave it as an exercise to show that there is a solution to $D_\mu \epsilon = 0$ for these solutions. In fact this metric appears frequently in String-Theory and M-theory and has many interesting features so it is worth understanding it in greater detail. Sadly we don't have time to do it justice here.

3.2.4 M2-branes

Maximal supergravities always contain form-fields. That is totally anti-symmetric tensors $C_{\mu_1 \ldots m_{p+1}}$ subject to an Abelian gauge transformation of the form $\delta C_{\mu_1,\ldots,m_{p+1}} = \partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_{p+1}]}$. This is a higher-form analogue of Maxwell's theory of electromagnetism, corresponding to $p = 0$. For concreteness let us concentrate on eleven-dimensional supergravity.

Let us look at another class of solution where the form field is non-zero. In particular lets start by imagining that C_{012} is non-zero. This breaks the spacetime into a three-dimensional Minkowskian part with coordinates $x^m, m = 0, 1, 2$ and an eight-dimensional Euclidean part with coordinates $x^I, I = 3, 4, ..., 10$. Therefore we start with an ansatz of the form

$$\begin{aligned} ds^2 &= e^{2A}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + e^{2B}((dx^3)^2 + ... + (dx^{10})^2) \\ G_{012I} &= -\partial_I C \end{aligned} \tag{94}$$

where $G_{\mu\nu\lambda\rho} = 4\partial_{[\mu}C_{\nu\lambda\rho]}$ and the functions A, B and C only depend on x^I. A vielbein frame is

$$\begin{aligned} e^{\underline{n}} &= e^A dx^n & e^{\underline{J}} &= e^B dx^J \\ de^{\underline{n}} &= \partial_I A e^{-B} e^{\underline{I}} \wedge e^{\underline{n}} & de^{\underline{J}} &= \partial_I B e^{-B} e^{\underline{I}} \wedge e^{\underline{J}} \\ &= \partial_{\underline{I}} A e^{\underline{I}} \wedge e^{\underline{n}} & &= \partial_{\underline{I}} B e^{\underline{I}} \wedge e^{\underline{J}} \end{aligned} \tag{95}$$

and you can check that

$$\omega^{\underline{n}}{}_{\underline{I}} = -\omega_{\underline{I}}{}^{\underline{n}} = \partial_{\underline{I}} A e^{\underline{n}} \qquad \omega^{\underline{I}}{}_{\underline{J}} = -\partial_{\underline{I}} B e^{\underline{J}} + \partial_{\underline{J}} B e^{\underline{I}} \tag{96}$$

satisfies

$$\begin{aligned} de^{\underline{n}} + \omega^{\underline{n}}{}_{\underline{m}} \wedge e^{\underline{m}} + \omega^{\underline{n}}{}_{\underline{J}} \wedge e^{\underline{J}} &= 0 \\ de^{\underline{I}} + \omega^{\underline{I}}{}_{\underline{m}} \wedge e^{\underline{m}} + \omega^{\underline{I}}{}_{\underline{J}} \wedge e^{\underline{J}} &= 0 \end{aligned} \tag{97}$$

We need to solve the supersymmetry variation equation $\delta\Psi_\mu = 0$ with

$$\delta\Psi_\mu = D_\mu\epsilon - \frac{1}{288}(\Gamma_\mu{}^{\nu\lambda\rho\sigma} - 8\delta^\nu_\mu\Gamma^{\lambda\rho\sigma})G_{\nu\lambda\rho\sigma} \tag{98}$$

Note that the only non-zero field strength is

$$G_{mnpI} = -\epsilon_{mnp}\partial_I C \tag{99}$$

What do we find:

$$\begin{aligned} D_m\epsilon &= \partial_m\epsilon + \frac{1}{2}\partial_I A e^{A-B}\Gamma_{\underline{mI}}\epsilon \\ &= \frac{1}{2}\partial_I A e^{A-B}\Gamma_{\underline{mI}}\epsilon \\ D_I\epsilon &= \partial_I\epsilon + \frac{1}{2}\partial_J B\Gamma_{\underline{IJ}}\epsilon \end{aligned} \tag{100}$$

where we assumed $\partial_m \epsilon = 0$ and

$$\begin{aligned}
\Gamma_m{}^{\nu_1 \dots \nu_4} G_{\nu_1 \dots \nu_4} \epsilon &= 0 \\
\Gamma_I{}^{\nu_1 \dots \nu_4} G_{\nu_1 \dots \nu_4} \epsilon &= 4 \Gamma_I{}^{mnpJ} G_{nmpJ} \\
&= -4! \Gamma_I{}^{012J} \partial_J C \epsilon \\
&= \mp 4! e^{-3A} \Gamma_{\underline{IJ}} \partial_J C \epsilon \\
\delta_m^{\nu_1} \Gamma^{\nu_2 \nu_3 \nu_4} G_{\nu_1 \nu_2 \nu_3 \nu_4} \epsilon &= -3 \epsilon_{mnp} \Gamma^{npI} \partial_I C \epsilon \\
&= -3! e^{-2A-B} \Gamma_{\underline{m}} \Gamma_{\underline{012}} \Gamma_{\underline{I}} \partial_I C \epsilon \\
&= \pm 3! e^{-2A-B} \Gamma_{\underline{m}} \Gamma_{\underline{I}} \partial_I C \epsilon \\
\delta_I^{\nu_1} \Gamma^{\nu_2 \nu_3 \nu_4} G_{\nu_1 \nu_2 \nu_3 \nu_4} \epsilon &= 3! \Gamma^{012} \partial_I C \epsilon \\
&= \pm 3! e^{-3A} \partial_I C \epsilon
\end{aligned} \tag{101}$$

where we have assumed $\Gamma_{\underline{012}} \epsilon = \pm \epsilon$.

From the $\delta \psi_m = 0$ equation we find

$$\begin{aligned}
\left[\frac{1}{2} \partial_I A e^{A-B} \Gamma_{\underline{mI}} \mp \frac{8 \cdot 3!}{288} e^{-2A-B} \Gamma_{\underline{m}} \Gamma_{\underline{I}} \partial_I C \right] \epsilon &= 0 \\
\frac{1}{2} e^{-2A-B} \Gamma_{\underline{m}} \Gamma_{\underline{I}} \left[\partial_I A e^{3A} \mp \frac{96}{288} \partial_I C \right] \epsilon &= 0
\end{aligned} \tag{102}$$

From which we learn that

$$C = \pm e^{3A} \tag{103}$$

Next we look at the $\delta \psi_I = 0$ equation

$$\begin{aligned}
\partial_I \epsilon + \frac{1}{2} \partial_J B \Gamma_{\underline{IJ}} \epsilon - \frac{1}{288} \left[\mp 4! e^{-3A} \Gamma_{\underline{IJ}} \partial_J C \mp 8 \cdot 3! e^{-3A} \partial_I C \right] \epsilon &= 0 \\
\partial_I \epsilon + \frac{1}{2} \partial_J B \Gamma_{\underline{IJ}} \epsilon - \frac{1}{288} \left[-3 \cdot 4! \Gamma_{\underline{IJ}} \partial_J A - 3 \cdot 8 \cdot 3! \partial_I A \right] \epsilon &= 0
\end{aligned} \tag{104}$$

where in the second line we have used $C = \pm e^{3A}$. The $\Gamma_{\underline{IJ}}$ terms cancel if

$$B = -\frac{2 \cdot 3 \cdot 4!}{288} A = -\frac{1}{2} A \tag{105}$$

which just leaves

$$\partial_I \epsilon + \frac{144}{288} \partial_I A \epsilon = \partial_I \epsilon + \frac{1}{2} \partial_I A \epsilon \tag{106}$$

which is solved by $\epsilon = e^{-\frac{1}{2}A} \epsilon_0$ with ϵ_0 constant and $\Gamma_{\underline{012}} \epsilon_0 = \pm \epsilon_0$

Lastly we need to check the 3-form equation of motion (the Einstein equations are typically automatically satisfied if $\delta \psi_\mu = 0$ - but feel free to check). One finds

$$\begin{aligned}
\star dC &= -\frac{1}{7!} \epsilon_{I_1 \dots I_7 K} e^{3A+8B} (e^{-2A})^3 e^{-2B} \partial_K (e^{3A}) dx^{I_1} \wedge \dots \wedge dx^{I_7} \\
&= -\frac{1}{7!} \epsilon_{I_1 \dots I_7 K} e^{-3A} \partial_K A dx^{I_1} \wedge \dots \wedge dx^{I_7} \\
d \star dC &= -\frac{1}{7!} \partial_L (e^{-3A} \partial_K A) \epsilon_{I_1 \dots I_7 K} dx^L \wedge dx^{I_1} \wedge \dots \wedge dx^{I_7} \\
&= -\partial_K (e^{-3A} \partial_K A) dx^3 \wedge \dots \wedge dx^{10}
\end{aligned} \tag{107}$$

from which we learn that $e^{-3A} = H$ is a harmonic function. If follows that $e^{2A} = H^{-\frac{2}{3}}$, $e^{2B} = H^{\frac{1}{3}}$ and $C = H^{-1}$:

$$\begin{aligned} ds^2 &= H^{-2/3}(-(dx^2)^2 + (dx^1)^2 + (dx^3)^2) + H^{1/3}((dx^3)^2 + ... + (dx^{10})^2) \\ C_{012} &= \pm H^{-1} \\ H &= 1 + \sum_{A=1}^{N} \frac{R^6}{|\underline{x} - \underline{x}_A|^6} \end{aligned} \tag{108}$$

Its easy to see that these solutions carry a charge with respect to the 3-form C:

$$\frac{1}{\mathrm{vol}(S^7)} \int_{S^7} \star G = \pm 6R^6 N \tag{109}$$

where the integral is over the spatial sphere at infinity in the eight-dimensional transverse space.

3.2.5 M5-branes

There is another natural solution where we make use of the Hodge-dual of dC:

$$\star dC = \tilde{G} \tag{110}$$

which is a 7-form and, on shell, is closed so $\tilde{G} = d\tilde{C}$ where $\tilde{C}$ is a 6-form[2]. Now we look for a solution of the form

$$\begin{aligned} ds^2 &= e^{2A}(-(dx^0)^2 + (dx^1)^2 + ... + (dx^5)^2) + e^{2B}((dx^6)^2 + ... + (dx^{10})^2) \\ G_{IJKL} &= \varepsilon_{IJKLM} \partial_M \tilde{C} \end{aligned} \tag{111}$$

where now $I, J = 6, 7, 8, 9, 10$ and again the functions A, B and C only depend on x^I. As before vielbein frame and spin connection are

$$e^{\underline{n}} = e^A dx^n \qquad\qquad e^{\underline{J}} = e^B dx^J \tag{112}$$

and

$$\omega^{\underline{n}}{}_{\underline{I}} = -\omega_{\underline{I}}{}^{\underline{n}} = \partial_{\underline{I}} A e^{\underline{n}} \qquad \omega^{\underline{I}}{}_{\underline{J}} = -\partial_{\underline{I}} B e^{\underline{J}} + \partial_{\underline{J}} B e^{\underline{I}} \tag{113}$$

all that changes is the range of the indices. So we find

$$\begin{aligned} D_m \epsilon &= \frac{1}{2} \partial_I A e^{A-B} \Gamma_{\underline{mI}} \epsilon \\ D_I \epsilon &= \partial_I \epsilon + \frac{1}{2} \partial_J B \Gamma_{\underline{IJ}} \epsilon \end{aligned} \tag{114}$$

[2] This isn't quite true, due to the $C \wedge dC \wedge dC$ term $\star dC - \frac{1}{2} C \wedge dC$ is closed but we won't need this subtly here.

and

$$\begin{aligned}
\Gamma_m{}^{\nu_1\ldots\nu_4}G_{\nu_1\ldots\nu_4}\epsilon &= e^{A-4B}\Gamma_{\underline{m}}\Gamma^{\underline{IJKL}}\epsilon_{IJKLM}\partial_M\tilde{C}\epsilon \\
&= 4!e^{A-4B}\Gamma_{\underline{m}}\Gamma_{\underline{M}}\Gamma_{\underline{678910}}\partial_M\tilde{C}\epsilon \\
&= \pm 4!e^{A-4B}\Gamma_{\underline{m}}\Gamma_{\underline{M}}\partial_M\tilde{C}\epsilon \\
\Gamma_I{}^{\nu_1\ldots\nu_4}G_{\nu_1\ldots\nu_4}\epsilon &= \Gamma_I{}^{JKLM}G_{JKLM}\epsilon \\
&= e^{-3B}\epsilon_{IJKLM}\Gamma_{\underline{678910}}\epsilon_{JKLMP}\partial_P\tilde{C}\epsilon \\
&= \mp 4!e^{-3B}\partial_I\tilde{C}\epsilon \\
\delta_m^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}G_{\nu_1\nu_2\nu_3\nu_4}\epsilon &= 0 \\
\delta_I^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}G_{\nu_1\nu_2\nu_3\nu_4}\epsilon &= \epsilon_{IJKLM}\Gamma^{JKL}\partial_M\tilde{C}\epsilon \\
&= 3!e^{-3B}\Gamma_{\underline{IM}}\Gamma_{\underline{678910}}\partial_M\tilde{C}\epsilon \\
&= \pm 3!e^{-3B}\Gamma_{\underline{IM}}\partial_M\tilde{C}\epsilon
\end{aligned} \tag{115}$$

where we now have assumed $\Gamma_{\underline{678910}}\epsilon = \pm\epsilon$. From the $\delta\psi_m = 0$ equation we find

$$\begin{aligned}
0 &= \left[\frac{1}{2}\partial_I A e^{A-B}\Gamma_{\underline{mI}} \mp \frac{4!}{288}e^{A-4B}\Gamma_{\underline{m}}\Gamma_{\underline{I}}\partial_I\tilde{C}\right]\epsilon \\
&= \frac{1}{2}e^{A-4B}\Gamma_{\underline{m}}\Gamma_{\underline{I}}\left[\partial_I A e^{3A} \mp \frac{96}{288}\partial_I\tilde{C}\right]\epsilon
\end{aligned} \tag{116}$$

From which we learn that

$$\partial_I\tilde{C} = \pm 6e^{3B}\partial_I A \tag{117}$$

Next we look at the $\delta\psi_I = 0$ equation

$$\begin{aligned}
\partial_I\epsilon + \frac{1}{2}\partial_J B\Gamma_{\underline{IJ}}\epsilon - \frac{1}{288}\left[\mp 4!e^{-3B}\partial_I C \mp 8\cdot 3!e^{-3B}\Gamma_{\underline{IJ}}\partial_J C\right]\epsilon = 0 \\
\partial_I\epsilon + \frac{1}{2}\partial_J B\Gamma_{\underline{IJ}}\epsilon - \frac{1}{288}\left[-6\cdot 4!\partial_I A - 6\cdot 8\cdot 3!\Gamma_{\underline{IJ}}\partial_J A\right]\epsilon = 0
\end{aligned} \tag{118}$$

where in the second line we have used $C = \pm 6e^{3B}\partial_I A$. The $\Gamma_{\underline{IJ}}$ terms cancel if

$$B = -\frac{12\cdot 8\cdot 3!}{288}A = -2A \tag{119}$$

which tells us that

$$\tilde{C} = \mp e^{-6A} \tag{120}$$

leaves us with

$$\partial_I\epsilon + \frac{144}{288}\partial_I A\epsilon = \partial_I\epsilon + \frac{1}{2}\partial_I A\epsilon \tag{121}$$

which is solved by $\epsilon = e^{-\frac{1}{2}A}\epsilon_0$ with ϵ_0 constant and $\Gamma_{\underline{678910}}\epsilon_0 = \pm\epsilon_0$

Lastly we need to check the 3-form equation of motion. In fact by construction we have $d\star G = 0$ however we don't have the Bianchi identity: $dG = 0$ it's easy to see that this means

$$\partial_I\partial_I\tilde{C} = \mp\partial_I\partial_I(e^{-6A}) = 0 \tag{122}$$

Thus $e^{-6A} = H$ is harmonic and so we find

$$\begin{aligned}
ds^2 &= H^{-1/3}(-(dx^2)^2 + ... + (dx^5)^2) + H^{2/3}((dx^6)^2 + ... + (dx^{10})^2) \\
\tilde{C} &= \mp H \\
H &= 1 + \sum_{A=1}^{N} \frac{R^3}{|\underline{x} - \underline{x}_A|^3}
\end{aligned} \tag{123}$$

Now we find that

$$\frac{1}{\mathrm{vol}(S^4)} \int_{S^4} G = \pm 3R^3 N \tag{124}$$

where the integral is over the spatial sphere at infinity in the five-dimensional transverse space.

3.3 Non-perturbative Stability

There is a method to establish that BPS solutions, that is ones that admit a spinor such that $\delta\psi_\mu = 0$ are classically stable. Namely one constructs the so-called generalised Nestor tensor (again things might be different in dimensions where spinors aren't real or there is more than just a gravitino):

$$E^{\mu\nu} = \bar{\epsilon}\Gamma^{\mu\nu\lambda}\delta\psi_\lambda \tag{125}$$

This has the property that, on-shell,

$$D_\mu E^{\mu\nu} = \overline{\delta\psi_\mu}\Gamma^{\mu\nu\lambda}\delta\psi_\lambda + \bar{\chi}\Gamma^\nu\chi \ . \tag{126}$$

for some χ. Note that in this section we take all spinors to be commuting classical variables.

Its a bit of a slog to show this for eleven-dimensional supergravity (try it! - Hint $\chi = 0$). However it is relatively easy to consider the case where only the metric is non-zero and hence $\delta\psi_\mu = D_\mu\epsilon$. Here we have

$$E^{\mu\nu} = \bar{\epsilon}\Gamma^{\mu\nu\lambda}D_\lambda\epsilon \tag{127}$$

so

$$\begin{aligned}
D_\mu E^{\mu\nu} &= D_\mu\bar{\epsilon}\Gamma^{\mu\nu\lambda}D_\lambda\epsilon + \bar{\epsilon}\Gamma^{\mu\nu\lambda}D_\mu D_\lambda\epsilon \\
&= \overline{\delta\psi_\mu}\Gamma^{\mu\nu\lambda}\delta\psi_\lambda + \frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\lambda}[D_\mu, D_\lambda]\epsilon \\
&= \overline{\delta\psi_\mu}\Gamma^{\mu\nu\lambda}\delta\psi_\lambda + \frac{1}{8}\bar{\epsilon}\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma}R_{\mu\lambda\rho\sigma}\epsilon
\end{aligned} \tag{128}$$

We have evaluated the second term before (see around (63)) and saw that it is proportional to $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ which, in the absence of matter, vanishes on-shell.

The Nestor tensor arose in a refinement of Witten's proof of the positive mass theorem in General Relativity. The idea is to compute the codimension-two integral

$$\oint E^{\mu\nu} dS_{\mu\nu} = \int D_\mu E^{\mu\nu} dS_\nu \tag{129}$$

We are always looking at spacelike surfaces so, on-shell,

$$\begin{aligned}
\int D_\mu E^{\mu\nu} dS_\nu &= \int D_i E^{i0} dS_0 \\
&= \int (\overline{\delta\psi_i}\Gamma^{i0j}\delta\psi_j + \bar{\chi}\Gamma^0\chi) dS_0 \\
&= \int (\delta\psi_i{}^T \Gamma^{ij}\delta\psi_j + \chi^T\chi) dS_0 \\
&= \int (\overline{\delta\psi_i}\Gamma^{i0j}\delta\psi_j + \bar{\chi}\Gamma^0\chi) dS_0 \\
&= \int (-(\Gamma^i\delta\psi_i)^T\Gamma^j\delta\psi_j + \delta\psi_i^T\delta\psi_i + \chi^T\chi) dS_0
\end{aligned} \tag{130}$$

The argument is that $\Gamma^i\delta\psi_i = 0$ is a Dirac-like equation of the form $\Gamma^i D_i\epsilon + \ldots = 0$ and it is always possible to find a solution with ϵ approaching a given constant spinor at spatial infinity. For such an ϵ we therefore have

$$\oint E^{i0} dS_{i0} = \int D_i E^{i0} dS_0 \geq 0 \tag{131}$$

On the other hand the left hand side is an integral over the sphere at spatial infinity and therefore doesn't depend on ϵ, just its asymptotic value ϵ_∞. Thus we obtain a bound

$$\oint \bar{\epsilon}_\infty \Gamma^{0ij}\delta\psi_j dS_{0i} \geq 0 \tag{132}$$

which is saturated if there is a solution to $\delta\psi_\mu = 0$, *i.e.* supersymmetric solutions.

Of course we need to see what the bound actually is. First lets look at the gravitational bit. Here we expanding the metric at spatial infinity as

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} + \ldots \\
e_\mu{}^{\underline{\nu}} &= \delta_\mu^{\underline{\nu}} + \frac{1}{2}h_{\mu\nu} + \ldots \\
\epsilon &= \epsilon_\infty + \ldots
\end{aligned} \tag{133}$$

where ϵ_∞ is constant.

$$\begin{aligned}
\bar{\epsilon}_\infty\Gamma^{0rj}D_j\epsilon &= i\bar{\epsilon}_\infty\Gamma^{0rj}(\partial_j\epsilon + \frac{1}{4}\omega_j{}^{\underline{\mu\nu}}\Gamma_{\underline{\mu\nu}}\epsilon) \\
&\sim \frac{i}{4}\bar{\epsilon}_\infty\Gamma^{0rj}\omega_j{}^{\underline{\mu\nu}}\Gamma_{\underline{\mu\nu}}\epsilon_\infty \\
&\sim \frac{ix^k}{4r}\bar{\epsilon}_\infty\Gamma^{0kj}\omega_k{}^{\underline{\mu\nu}}\Gamma_{\underline{\mu\nu}}\epsilon_\infty
\end{aligned} \tag{134}$$

where we used the fact that the $\partial_i\epsilon$ term is subleading (at least for solutions that are spherically symmetric at large r) and doesn't contribute to the integral as as $r\to\infty$. To continue we assume that $\omega_k{}^{\underline{0j}}=0$ or at least is subleading. We then have

$$\bar\epsilon_\infty\Gamma^{0rj}D_j\epsilon\sim\frac{ix^k}{4r}\bar\epsilon_\infty\Gamma^{\underline{0kj}}\omega_j{}^{\underline{mn}}\Gamma_{\underline{mn}}\epsilon_\infty \tag{135}$$

where we use the fact that at infinity the Γ-matrices become those of flat space. Using the formula (26) we find, to leading order,

$$\begin{aligned}\omega_j{}^{mn}&=\delta^{l[m}\partial_{[j}h_{l]}{}^{n]}-\frac{1}{2}\delta^{l[m}\delta^{n]s}\delta_{jt}\partial_l h_s{}^t\\&=\frac{1}{4}\partial_jh_{mn}-\frac{1}{4}\partial_jh_{nm}-\frac{1}{4}\partial_mh_{nj}+\frac{1}{4}\partial_nh_{mj}-\frac{1}{4}\partial_mh_{nj}+\frac{1}{4}\partial_nh_{mj}\\&=-\frac{1}{2}\partial_mh_{nj}+\frac{1}{2}\partial_nh_{mj}\end{aligned} \tag{136}$$

Next we note that

$$\Gamma^{kj}\Gamma^{mn}=\Gamma^{kjmn}+\delta^{jm}\Gamma^{kn}-\delta^{km}\Gamma^{jn}+\delta^{kn}\Gamma^{jm}-\delta^{jn}\Gamma^{km}+\delta^{jm}\delta^{kn}-\delta^{km}\delta^{jn} \tag{137}$$

The first term drops out as it is anti-symmetric in j and n and j and m. Continuing we find

$$\begin{aligned}\Gamma^{kj}\Gamma^{mn}\omega_j{}^{mn}=&-\frac{1}{2}\partial_jh_{nj}\Gamma^{kn}+0-\frac{1}{2}\partial_mh_{nj}\Gamma^{jm}+\frac{1}{2}\partial_mh_{nn}\Gamma^{km}\\&+\frac{1}{2}\partial_nh_{mm}\Gamma^{kn}-\frac{1}{2}\partial_nh_{kj}\Gamma^{jn}-0+\frac{1}{2}\partial_nh_{mn}\Gamma^{km}\\&-\partial_mh_{mk}+\partial_kh_{nn}\end{aligned} \tag{138}$$

The terms with Γ's in them will vanish in the integral due to rotational symmetry as the leading order metric term is just a function of r (*i.e.* one will only find terms of the form $x^mx^k\Gamma^{km}$)

Thus we find

$$\int\bar\epsilon_\infty\Gamma^{0rj}D_j\epsilon=\frac{1}{2}\epsilon^T_\infty\epsilon_\infty\int\frac{x^k}{2r}(\partial_kh_{nn}-\partial_mh_{mk})d^{D-2}x \tag{139}$$

The integral is known as the ADM mass (energy) of a spacetime.

The remaining terms in $\delta\psi_\mu$ will give charges. For concreteness we consider eleven-dimensional supergravity where

$$\delta\psi_\mu=D_\mu\epsilon-\frac{1}{288}\left(\Gamma_\mu{}^{\nu_1..\nu_4}-8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)G_{\nu_1..\nu_4} \tag{140}$$

Let us consider the M2-brane case where $G_{012K}\neq0$ at infinity. Now we are only interested in the sphere at transverse infinity: off the M2-brane. Thus our i,j indices don't take the values $1,2$ but $I,J=3,..,10$. We are really calculating the mass per unit area in the (x^1,x^2)-plane. We also note that $\Gamma^r=\frac{x^K}{r}\Gamma^K$. Thus the extra contribution we are looking at is

$$
\begin{aligned}
\frac{1}{288}\Gamma^{0rJ}\left(\Gamma_J{}^{\nu_1..\nu_4}-8\delta_J^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)G_{\nu_1..\nu_4}\epsilon_\infty &= \frac{1}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{0KJ}\left(4!\Gamma_J{}^{012L}-8\cdot 3!\delta_J^L\Gamma^{012}\right)G_{012L}\epsilon_\infty \\
&= \frac{4!}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{KJ}\Gamma^{12}\left(\Gamma_{JL}+2\delta_J^L\right)G_{012L}\epsilon_\infty \\
&= \frac{4!}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{KJ}\Gamma^{12}\left(\Gamma_J\Gamma_L+\delta_J^L\right)G_{012L}\epsilon_\infty \\
&= \frac{4!}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{12}\left(6\Gamma^K\Gamma_L+3\Gamma^{KL}\right)G_{012L}\epsilon_\infty
\end{aligned}
\tag{141}
$$

Note that since we are at infinity we can use the flat space Γ-matrices. Now near infinity we have $G_{012K}=\frac{x^K}{r}G_{012r}$ and hence

$$
\begin{aligned}
\frac{1}{288}\Gamma^{0rJ}\left(\Gamma_J{}^{\nu_1..\nu_4}-8\delta_J^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)G_{\nu_1..\nu_4}\epsilon_\infty &= \frac{6\cdot 4!}{288}\bar{\epsilon}_\infty\Gamma^{12}G_{012r}\epsilon_\infty \\
&= \pm\frac{1}{2}\epsilon_\infty^T\epsilon_\infty G_{012r}
\end{aligned}
\tag{142}
$$

Thus we simply find the M2-brane charge. A similar calculation for the M5-brane shows that

$$
\begin{aligned}
\frac{1}{288}\Gamma^{0rJ}\left(\Gamma_J{}^{\nu_1..\nu_4}-8\delta_J^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)G_{\nu_1..\nu_4}\epsilon_\infty &= \frac{1}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{0KJ}\left(\Gamma_J{}^{LMNP}-8\delta_J^L\Gamma^{MNP}\right)G_{LMNP}\epsilon_\infty \\
&= \frac{1}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{0KJ}\Gamma_{\underline{678910}}\left(\epsilon_{JLMNP}-4\delta_J^L\epsilon_{MNPRS}\Gamma^{RS}\right)\epsilon_{LMNP} \\
&= \frac{1}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^{0KJ}\Gamma_{\underline{678910}}\left(4!\partial_J\tilde{C}-2\cdot 4!\Gamma^{JQ}\partial_Q\tilde{C}\right)\epsilon_\infty
\end{aligned}
\tag{143}
$$

At infinity the first term vanishes as $\partial_J\tilde{C}\sim x^J\partial_r\tilde{C}$ and from the second term we find

$$
\begin{aligned}
\frac{1}{288}\Gamma^{0rJ}\left(\Gamma_J{}^{\nu_1..\nu_4}-8\delta_J^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)G_{\nu_1..\nu_4}\epsilon_\infty &= -\frac{2\cdot 4!}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^0\Gamma_{\underline{678910}}\Gamma^{KJ}\left(\Gamma^J\Gamma^Q-\delta^{JQ}\right)\partial_Q\tilde{C}\epsilon_\infty \\
&= -\frac{2\cdot 4!}{288}\bar{\epsilon}_\infty\frac{x^K}{r}\Gamma^0\Gamma_{\underline{678910}}\left(3\Gamma^K\Gamma^Q-\Gamma^{KQ}\right)\partial_Q\tilde{C}\epsilon_\infty \\
&= \pm\frac{1}{2}\epsilon_\infty^T\epsilon_\infty\partial_r\tilde{C}
\end{aligned}
\tag{144}
$$

which just measures the M5-brane charge. So the bound we get says that the ADM mass of the spacetime is bounded below by the charges with equality iff there is supersymmetry. That is to say no classical process can lower the mass of the spacetime below the bound set by the charges and the charges are themselves conserved.

3.4 Branes more generally

We have found 4 types of supersymmetric solutions to eleven-dimensional supergravity: plane gravitational wave, Multi-Taub-NUT, M2-branes and M5-branes. By dimension reduction these lead to solutions of type IIA supergravity.

Let us first look at the two brane solutions we found. These both carry charges with respect to the 4-form field $G = dC$. In particular

$$
\begin{aligned}
Q_{M2} &= \frac{1}{vol(S^7)} \int_{S^7} \star G \\
Q_{M5} &= \frac{1}{vol(S^4)} \int_{S^7} G
\end{aligned}
\tag{145}
$$

The M2-brane is said to have an electric charge and the M5-brane a magnetic charge with respect to G. The spheres are the spheres at spatial infinity in the transverse space to the plane where the singularity sits. This follows from a higher-dimensional version of electromagnetic duality.

More generally if we have an $(p+2)$-form field strength F_{p+2} that is **closed**, meaning $dF_{p+1} = 0$ then we can (at least locally) find a potential A_{p+1} such that $F = dA_{p+1}$. This often occurs by definition and the equation $dF_{p+1} = 0$ is automatic due to the fact that $d^2 = 0$ and is referred to as a **Bianchi identity**. But as we saw in the case of the $M5$-brane this was not automatic. One then needs to solve the equation of motion which is typically of the form $d \star F_{p+2} = d \star dA_{p+1} = 0$. Such a potential naturally couples to an object with an $(p+1)$-dimensional worldvolume:

$$
S_{electric} = T_p \int_{\Sigma_{p+1}} {}^*A \tag{146}
$$

Here Σ_{p+1} is a $(p+1)$-dimensional surface in spacetime which includes the time direction and T_p is a constant with dimensions of $(mass)^{p+1}$. The star here indicates the pull-back of A_{p+1} to Σ:

$$
{}^*A_{m_1 \ldots m_{p+1}} = \partial_{m_1} X_1^{\mu} \ldots \partial_{m_{p+1}} X^{\mu_{p+1}} A_{\mu_1 \ldots \mu_{p+1}}(X(\sigma)) \tag{147}
$$

where σ^m are local coordinates on Σ_{p+1} and $X^\mu : \Sigma_{p+1} \to \mathcal{M}_D$ are the embedding coordinates of Σ_{p+1} into the bulk D-dimensional spacetime.

So we can think of it as the worldvolume of a p-dimensional object. Such an object is known as a p-brane and this coupling is called an electric coupling.

On the other hand the equations of motion for F_{p+2} are of the form $d \star F_{p+2} = 0$ (perhaps with additional terms that we can ignore here). An alternative way to solve this equation is to write $F_{p+2} = \star d\tilde{A}_{D-p-3}$ for some form $\tilde{A}_{D-p-3}$. In this case the Bianchi identity turns into $d \star d\tilde{A}_{D-p-3} = 0$. This means that there is also a coupling to F_{p+2} of the form

$$
S_{magnetic} = \tilde{T}_{D-p-4} \int_{\Sigma_{D-p-3}} {}^*\tilde{A} \tag{148}
$$

which is called a magnetic coupling. Thus we find a natural generalization of four-dimensional electromagnetic duality. In particular if we set $D = 4$ and $p = 0$ then F_{p+2} is the two-form of Maxwell's theory, A is the usual 1-form potential and $\tilde{A}$ a magnetic dual 1-form. We see that supergravities which have a variety of form-fields then naturally admit states that carry electric or magnetic charges. However in general these will be p-branes and $(D-p-4)$-branes.

These two objects are mutually non-local in the sense that their potentials are related by

$$dA_{p+1} = \star d\tilde{A}_{D-p-3} \tag{149}$$

Just as in four-dimensions we find a Dirac quantization condition that ensures the Dirac-string that appears in the magnetic solution is unobservable:

$$(2\pi)^{D-4} l^{D-2} T_p \tilde{T}_{D-p-4} \in \mathbb{Z} \tag{150}$$

where l is the Planck length. Lastly it should be mentioned that the choice of sign above corresponds to p-branes, with positive charges and anti-p-branes with negative charges.

We have seen in all the solutions that one finds a harmonic function which admits an arbitrary number of simple poles. Indeed the solutions are essentially the same as four-dimensional extreme Riessner-Nordstrom solutions of Einstein-Maxwell theory. Each such pole contributes one to the charge and in this way we see that we can allow for an arbitrary number N of branes. How is this possible? Well their gravitational attraction is exactly cancelled by the repulsive force arising from the form field.

3.5 Branes in String Theory

Since the type IIA supergravity arises from dimensional reduction of eleven-dimensional supergravity on S^1 we can take the four solutions we found above and map them to solutions of type IIA supergravity. They must then have interpretations as descriptions of states in type IIA string theory. So what do we find (in string frame)?

Reduction of the pp-wave along x^1 yields the following solution (in this section H is always a Harmonic function on the transverse space):

$$\begin{aligned} ds^2 &= -H^{-\frac{1}{2}}(dx^0)^2 + H^{\frac{1}{2}}((dx^1)^2 + ...(dx^9)^2) \\ A_0 &= \mp H^{-1} \\ e^\phi &= H^{\frac{3}{4}} \end{aligned} \tag{151}$$

This is a black hole solution and is called the D0-brane. Note that I have relabelled the coordinates also that the H here is equal to $1 + H_{there}$ where H_{there} is what appears in (85) and one normally imposes $H_{there} \to 0$ at infinity).

Reduction of the multi-Taub-NUT metric along x^1 yields

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0) + (dx^1)^2 + ...(dx^5)^2) + H^{\frac{1}{2}}((dx^7)^2 + (dx^8)^2 + (dx^9)^2) \\ \tilde{A}_{0123456} &= \mp H^{-1} \\ e^\phi &= H^{-\frac{3}{4}} \end{aligned} \tag{152}$$

This is a black 6-brane solution known as the D6-brane. In ten-dimensions is is electro-magnetically dual to the D0-brane solution above.

Reducing the M2-brane solution gives two different ten-dimensional solutions depending on where we take the eleventh dimension to lie. If is is along the M2-brane we find

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}}(-(dx^0)^2 + (dx^1)^2) + H^{\frac{2}{3}}((dx^2)^2 + ...(dx^9)^2) \\ B_{01} &= \mp H^{-1} \\ e^\phi &= H^{-\frac{1}{2}} \end{aligned} \tag{153}$$

This is a string-like solution and we interpret it as the effective gravitational field profile away from a fundamental string of type IIA string theory. On the other hand reducing along a direction transverse to the M2-brane gives

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + H^{\frac{1}{2}}((dx^3)^2 + ...(dx^9)^2) \\ C_{012} &= \mp H^{-1} \\ e^\phi &= H^{1/4} \end{aligned} \tag{154}$$

giving the D2-brane solution.

Reducing the M5-brane solution also gives two different ten-dimensional solutions depending on where we take the eleventh dimension to lie. If it is along the M5-brane we find

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + ... + (dx^4)^2) + H^{\frac{1}{2}}((dx^5)^2 + ...(dx^9)^2) \\ \tilde{C}_{01234} &= \mp H^{-1} \\ e^\phi &= H^{-1/4} \end{aligned} \tag{155}$$

This is a D4-brane solution. On the other hand reducing along a direction transverse to the M5-brane gives

$$\begin{aligned} ds^2 &= (-(dx^0)^2 + ... + (dx^5)^2) + H((dx^6)^2 + ...(dx^9)^2) \\ \tilde{B}_{012345} &= \mp H^{-1} \\ e^\phi &= H^{\frac{1}{2}} \end{aligned} \tag{156}$$

giving solution known as the NS5-brane. This solution is the magnetic dual to the fundamental string solution above. It is known to correspond to an exact worldsheet superconformal field theory (exact meaning valid to all orders in α').

Thus we have found the fundamental string as a solution to type IIA supergravity but also its magnetic dual known as the NS5-branes and then D0, D2, D4 and D6-brane solutions. These latter solutions are known to correspond to Dp-branes which are $(p+1)$-dimensional planes in spacetime where open strings can end. Although these states are non-perturbative in string theory, their tensions scale as $1/g_s$, their dynamics is perturbative and described by open string diagrams. Their low energy dynamics is governed by $U(N)$ maximally supersymmetric Yang-Mills

3.5.1 p-branes in type IIB Supergravity

Both the fundamental string solution and its magnetic dual the NS5-brane only involve the metric, dilaton and Kalb-Ramond B-field. As such they are universal solutions to all ten-dimensional supergravities, including type IIB.

If we look at the Dp-brane solutions ($p = 0, 2, 4, 6$) we found above we see that they have a universal formula (in string frame):

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-(dx^0)^2 + (dx^1)^2 + ... + (dx^p)^2) + H^{\frac{1}{2}}((dx^{p+1})^2 + ...(dx^9)^2) \\ A_{01...p} &= \pm H^{-1} \\ e^{\phi} &= H^{-(p-3)/4} \end{aligned} \tag{157}$$

As we have seen the solutions for even p arise in type IIA supergravity (the case of $p = 8$ is special and does indeed appear in ten-dimensions but has no eleven-dimensional description). None of these are solutions to type IIB supergravity as there aren't form field of the right rank: type IIB has even-form fields. However it turns out that type IIB does have Dp-brane solutions for $p = 1, 3, 5, 7$ (and even $p = -1$!). These can be obtained from type IIA string theory by T-duality which we will mention shortly.

4 Reduction on Tori and U-duality

A remarkable thing happens when you reduce maximal supergravities on tori to obtain lower-dimensional theories. Namely rather exotic and unexpected symmetry groups arise. One naturally expects that reduction on an n-dimensional torus would lead to an $SL(n, \mathbb{R})$ symmetry from the symmetries of the torus. Thus eleven-dimensional supergravity compactified to eight-dimensions on a three-torus will have an $SL(3, \mathbb{R})$ symmetry. However we have seen that the type IIB supergravity has an $SL(2, \mathbb{R})$ already in ten dimensions. Reducing it to eight dimensions on a two-torus preserves this but will include another $SL(2, \mathbb{R})$ so we expect an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry. Unlike in ten dimensions the maximal supergravities in lower dimensions are unique so whether we compactify eleven-dimensional supergravity on a three-torus or type IIB supergravity on a two-torus we get the same theory. However one has $SL(3, \mathbb{R})$ symmetry and the other $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. So it turns out that the eight-dimensional theory has

$SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$ symmetry! Further reduction leads to a curious list of symmetries:

$$\begin{array}{lccc}
D=11 & & \emptyset & \\
D=10 & IIA:\emptyset & & IIB:SL(2) \\
D=9 & & SL(2) & \\
D=8 & & SL(2)\times SL(3) & \\
D=7 & & SL(5) & \\
D=6 & & SO(5,5) & \\
D=5 & & E_6 & \\
D=4 & & E_7 & \\
D=3 & & E_8 &
\end{array} \tag{158}$$

This enhancement is similar in spirit to Ehlers symmetry. This arises in four-dimensional pure gravity compactified on a circle. The four-dimensional metric components lead to a metric, vector A and scalar ϕ in three-dimensions. However in three-dimensions the vector A can be dualized into a scalar a ($dA = \star_3 da$). Thus the action can be written as three-dimensional gravity coupled to two scalar fields which again we combine into $\tau = a + ie^{-\phi}$. However miraculously one finds

$$S_{3D\ Gravity} = \frac{1}{\kappa^2}\int\left(\frac{1}{2}R\star 1 + \frac{1}{2}\frac{\star d\tau\wedge d\tau}{(\tau-\bar{\tau})^2}\right) \tag{159}$$

What is remarkable is that this action has an $SL(2,\mathbb{R}$ symmetry acting as fractional linear transformation on τ (just as in type IIB above). One can use it as a solution generating technique since reduction on a circle merely requires that there is a Killing direction. Using this one can, for example, map the Schwarzschild solution to (Minkowsking) Taub-NUT.

In String Theory the reduced actions take the form (in Einstein frame)[3]

$$S = \frac{1}{\kappa^2}\int\left(\frac{1}{2}R\star 1 + \frac{1}{2}\mathrm{tr}(\star g^{-1}\mathcal{D}gg^{-1}\mathcal{D}g) + \sum_{p=1}^{D/2} G^{(p)}_{IJ}\star d\mathcal{A}^I_{p+1}\wedge d\mathcal{A}^I_{p+1}\right) \tag{160}$$

here g is an element of the coset G/H where G is the duality group and H is maximally compact subgroup. The covariant derivative $g^{-1}\mathcal{D}g$ is the projection of the usual derivative on to the components of the Lie algebra orthogonal to the Lie algebra of H. In addition $G^{(p)}_{IJ}$ is some metric on the set of $(p+1)$-forms labelled by I. The range of I is different for each p but the $(p+1)$-forms form a representation of G (for example in five-dimensions there are 27 one-forms corresponding to the fundamental representation of E_6).

These theories contain various p-branes that carry charges with respect to the various form fields. Under these symmetry groups the charges of the various p-brane rotate into each other. Since charges are quantized the continuous groups that we find in supergravity cannot be symmetries but discrete subgroups of them are. For example in type IIB supergravity the $SL(2,\mathbb{R})$ becomes $SL(2,\mathbb{Z})$. By looking at higher derivative terms in supergravity one sees that only a discrete subgroup can survive.

[3] This isn't quite the full story as extra care needs to be taken when $p+2 = D/2$.

4.1 U-Duality and M-Theory

String theory is formulated in terms of a perturbatively finite expansion using the string worldsheet. There are five consistent such expansions:

type IIA, type IIB, type I $SO(32)$, Heterotic $SO(32)$, Heterotic $E_8 \times E_8$

Each of these has a low energy effective action that is a supergravity. Indeed the first two supergravities share the same name as the string theory, no coincidence there. The final three have only one ten-dimensional supersymmetry (with corresponding components) but they also posses a ten-dimensional gauge field. Green-Schwarz anomaly cancellation restricts the choice to $SO(32)$ or $E_8 \times E_8$. The supergravities for these cases are also determined uniquely at two-derivative order given their gauge group. So the list of consistent ten-dimensional supergravities is

type IIA, type IIB, type I $SO(32)$, type I $E_8 \times E_8$

Note that there are only four supergravities but five string theories! Nobody thought much about this at the time as supergravities were just a low energy approximation. And certainly if you compute perturbative string scattering amplitudes you find different answers. But supergravities have one thing going in their favour over string theory: they are exact in the string coupling constant. Thus in cases of weak curvature supergravity solutions are exact in g_s.

So what's up? It turns out that two of the string theories are non-peturbatively equivalent. It can only be the type I and Heterotic $SO(32)$ strings. However the map between them is an S-duality in that it interchanges strong and weak coupling" $g_{type\ I} = 1/g_{het}$. Thus their equivalence can't be seen in the perturbative expansion.

In fact there is a web of dualities. Type IIB string theory is mapped to itself under S-duality which takes $g_{IIB} \to 1/g_{IIB}$. Upon reduction on a single S^1 type IIA and type IIB strings become equivalent via T-duality with is a known exact symmetry of String Theory perturbatively. So too do their supergravities: hence there is a unique maximally supersymmetric supergravity in each dimension below ten. Similarly so do the two Heterotic theories as a generic Wilson line will break their gauge groups to $U(1)^{16}$. Lastly we have that Type IIA string theory on $K3$ (this is a single family of four-dimensional compact manifolds which admit a single covariantly constant spinor η and have holonomy $SU(2) \subset SO(4)$) is the same as Heterotic string theory (either one) on $\mathbb{T}^4$.

There is one last player. We have seen that type IIA supergravity arises from dimensional reduction of eleven-dimensional supergravity. There is no known microscopic theory underlying eleven-dimensional supergravity as there is for ten-dimensional supergravities. But we believe it exists and goes by the name of M-theory. The key idea is that that the D0-branes provide a KK-like spectrum of states with masses and momenta

$$M = \frac{|n|}{g_s} \qquad P_{11} = \frac{n}{g_s} \qquad n \in \mathbb{Z} \tag{161}$$

and identifying $g_s = R_{11}/l_s$ where R_{11} is the radius of the extra-dimension the type IIA string theory will non-perturbatively regrow the eleventh dimension that we discarded to construct it. Indeed string perturbation theory is an expansion about $R_{11} = 0$. Furthermore the Heterotic $E_8 \times E_8$ string theory arises from reducing M-theory on an interval. The duality above between Type IIA on K3 and Heterotic Strings on T^4 now lifts to M-theory on K3 and Heterotic Strings on T^3.

There is much evidence for M-theory but its BPS states are, as we have seen, M2-branes and M5-branes and these cannot be used to construct a consistent worldsheet expansion that we see in String theory. Although using the AdS/CFT correspondence they can give gravitational duals to M-theory in certain asymptotically AdS spacetimes (but we don't know about AdS/CFT in these lectures).

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