LonTI 2021:
Gravity and black holes in AdS

Toby Wiseman (Imperial College)

Useful references:

There are many papers and reviews covering aspects of this. I co-wrote one some time back that might be useful for more details;

`arXiv:1312.0612` Holographic thermal field theory on curved spacetimes (D. Marolf, M. Rangamani, TW)

For lecture notes on stress tensors in asymptotic AdS spacetimes and renormalization of the on-shell action see;

`hep-th/0209067` Lecture notes on holographic renormalization (K. Skenderis)

For the following problem sheet do as many questions as you find useful, they get gradually harder...

Problem sheet:

1. Compute the conformal boundary geometry and boundary stress tensor for $(d+1)$-dimensional AdS in the Poincare chart, so,

$$ds^2 = \frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)$$

for any dimension d. You should find the (conformal) boundary geometry is Minkowski and the stress tensor vanishes. (Note that since the boundary is flat, local curvature invariants all vanish.)

2. Compute the same quantities for global AdS for the cases $d = 3$ and 4;

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2_{(d-1)} \ , \quad f = \frac{r^2}{\ell^2} + 1$$

Hence compute the energy of the global AdS spacetime, and show it vanishes for $d = 3$ but does not for $d = 4$ (in AdS-CFT this may be interpreted as a Casimir energy). You may find the coordinate $r = \left(\frac{\ell^2}{z} - \frac{z}{4}\right)$ useful.

3. Consider the planar AdS-Schwarzschild spacetime;

$$ds^2 = \frac{\ell^2}{z^2}\left(-f dt^2 + dx^a dx^a + \frac{1}{f}dz^2\right) \ , \quad f = 1 - \left(\frac{z}{z_0}\right)^d$$

with coordinates (t, x^a, z). Let us assume the $(d-1)$ spatial coordinates x^a are compactifed to form a torus with periods L, so $x^a \sim x^a + L$.

What is the (conformal) boundary geometry and stress tensor? Calculate the energy and the pressure of the spacetime.
Hint: you may find the following coordinate u useful; $z = u\left(1 - \frac{1}{2d}\left(\frac{u}{z_0}\right)^d + \ldots\right)$

Compute the temperature (wrt to $\partial/\partial t$) using Euclidean techniques.

You should find that the energy in terms of temperature is: $\frac{E}{c_{eff}} = \frac{(d-1)}{L}\left(\frac{4\pi T L}{d}\right)^d$.

Use the First Law to deduce the entropy of the solution. Compute the entropy from horizon area, $S = A/(4G)$, and confirm you get the same result.

4. For global AdS-Schwarzschild;

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2_{(d-1)} \ , \quad f = \frac{r^2}{\ell^2} + 1 - \left(\frac{r_0^2}{\ell^2} + 1 \right) \left(\frac{r_0}{r} \right)^{d-2}$$

compute the temperature of the spacetime. Show there is a minimum temperature for these black holes.

Compute the entropy. Hence deduce the difference of the energy to that of global AdS. (Do this without using the stress tensor, as in general dimension this is complicated.)
Hint: try computing the temperature and entropy and using the First Law.

Hence show that the specific heat capacity changes sign and becomes positive for $r_0 > \ell \sqrt{\frac{d-2}{d}}$. Black holes with positive specific heat are called 'large' black holes, the ones with negative specific heat are known as 'small' black holes.

Consider the free energy $F = E - TS$. Show that at fixed temperature (ie. the canonical ensemble) there is a 1st order phase transition – the Hawking-Page transition – between global AdS and the large black holes.

5. When the boundary spatial geometry is a product of time with a torus (as above $x^a \sim x^a + L$) a smooth bulk geometry is given by the 'AdS-Soliton';

$$ds^2 = \frac{\ell^2}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + f d\theta^2 + \frac{1}{f} dz^2 \right) \ , \quad f = 1 - \left(\frac{z}{z_s} \right)^d$$

where $x^\mu = (t, x^i)$ with $i = 1, \ldots, (d-2)$. The constant z_s is fixed in terms of L by the requirement that the spacetime is smooth. Compute this constant.

This spacetime has no horizon, and hence zero entropy. Compute the energy and pressure from the boundary stress tensor.

Show that this AdS-Soliton is preferred energetically over Poincare-AdS (compactified on a spatial torus). Further show that in the canonical ensemble (fixed temperature) there is a first order phase transition from this AdS-Soliton to the (toroidally compactified) planar AdS-Schwarzschild black hole.

6. Consider again the global AdS-Schwarzschild spacetime for $d = 3$. We may evaluate the free energy using $F = -T \ln Z$ where Z is the Euclidean gravitational partition function, which in the saddle point approximation is,

$$Z \simeq e^{-S_{Euc}} \ .$$

Here S_{Euc} is the renormalized Euclidean on-shell action given by the regulated action 'cutting off' the conformal boundary by only taking $z \geq \epsilon$ where z is a Fefferman-Graham coordinate, and the regulated action is,

$$S_{Euc} = -\frac{1}{8\pi G} \int_{z \geq \epsilon} d^4x \sqrt{g} \left(\frac{R}{2} - \Lambda \right) - \frac{1}{8\pi G} \int_{z=\epsilon} d^3x \sqrt{h} K + a \int_{z=\epsilon} d^3x \sqrt{h} + b \int_{z=\epsilon} d^3x \sqrt{h} R$$

where the last two terms are required to renormalize the action as you remove the regulator taking $\epsilon \to 0$, and a, b are constants you should determine. Compute this to confirm you obtain the same expression for the free energy that you obtained in Qu 4.