Supersymmetry, complex geometry and the hyperkähler quotien

Ulf Lindström[1]

[1]Leverhulme visiting professor to Imperial College

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Sigma models

$$\phi^\mu : \Sigma \to \mathcal{T} \quad : \quad x \mapsto \phi^\mu(x)$$

$$\int dx \left(g_{\mu\nu}(\phi)\partial\phi^\mu\partial\phi^\nu + \dots \right)$$

$$g_{\mu\nu}(\phi)\left(\partial^2\phi^\nu + \partial\phi^\rho\Gamma_{\rho\kappa}^{\ \ \nu}\partial_\mu\phi^\kappa + \dots\right) = 0$$

The geometry on $\mathcal{T}$ will depend on the symmetries of the model and the dimension of Σ.

A general bosonic sigma model in $2d$ is

$$\int d^2x \partial_{+\!\!+} \phi^\mu (g_{\mu\nu} + b_{\mu\nu})(\Phi)(\phi) \partial_= \phi^\nu + ...$$

$$=: \int d^2x \partial_{+\!\!+} \phi^\mu E_{\mu\nu}(\phi) \partial_= \phi^\nu + ...$$

$$\int d^2x \left(g_{\mu\nu}(\phi) d\phi^\mu \wedge {}^* d\phi^\nu + b_{\mu\nu}(\phi) d\phi^\mu \wedge d\phi^\nu \right)$$

$$=$$

$$\int d^2x \left(g_{\mu\nu}(\phi) \eta \partial\phi^\mu \partial\phi^\nu + b_{\mu\nu}(\phi) \epsilon \partial\phi^\mu \partial\phi^\nu \right)$$

Field equations from $\delta\phi^\lambda$:

$$0 = g_{\lambda\kappa} \partial^2 \phi^\kappa + \overbrace{\tfrac{1}{2}(g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})}^{\Gamma^0} \partial\phi^\mu \partial\phi^\nu$$

$$+ \underbrace{\tfrac{1}{2}(b_{\mu\lambda,\nu} + b_{\lambda\nu,\mu} - b_{\mu\nu,\lambda})}_{\frac{1}{2}H = T} \epsilon \partial\phi^\mu \partial\phi^\nu$$

Isometries 1.

There are several contexts where isometries of the target space metric and b field are important, e.g., when constructing new geometries via quotients, or for T-duality. Isometries are transformations of the target space coordinates X^μ that leave the metric and H field invariant. They can act with or without fixed points (freely) and be realised as translations (no fixed point) or rotations via matrix action etc. We shall assume that the infinitesimal transformations of the coordinates may be written

$$\delta X^\mu = \epsilon^{\mathbb{A}} k^\mu_{\mathbb{A}}(X) \ ,$$

with constant parameters $\epsilon^{\mathbb{A}}$, and that the infinitesimal generators $k_{\mathbb{A}} = k^\mu_{\mathbb{A}}\partial_\mu \ , \ \mathbb{A} = 1, ..., d$, form an d dimensional algebra

$$[k_{\mathbb{A}}, k_{\mathbb{B}}] = f_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} k_{\mathbb{C}} \ ,$$

where $[,]$ is the Lie bracket

Isometries 2.

The isometry requirement becomes

$$\mathcal{L}_{k_{\mathbb{A}}} g_{\mu\nu} = 0 \ , \quad \mathcal{L}_{k_{\mathbb{A}}} H_{\mu\nu\rho} = 0 \ ,$$

where the last expression may be written

$$\begin{aligned} &\mathcal{L}_{k_{\mathbb{A}}} H = i_{k_{\mathbb{A}}} dH + d(i_{k_{\mathbb{A}}} H) = d(i_{k_{\mathbb{A}}} H) = 0 \\ &\Rightarrow i_{k_{\mathbb{A}}} db = dv_{\mathbb{A}} \\ &\Rightarrow \mathcal{L}_{k_{\mathbb{A}}} b = d(v_{\mathbb{A}} + i_{k_{\mathbb{A}}} b) \end{aligned}$$

locally, for some one-form $v_{\mathbb{A}}$ defined up to the addition of an exact one-form. Note that only when $v_{\mathbb{A}} = i_{k_{\mathbb{A}}} b$ is the Lie derivative of b zero. We set $v_{\mathbb{A}} = 0$ below.

Gauging Isometries 1.

When $\epsilon^{\mathbb{A}} = \epsilon^{\mathbb{A}}(x)$ the action

$$S = \int d^2x \partial_{+\!\!+} X^\mu g_{\mu\nu} \partial_{=} X^\nu$$

is no longer invariant. Introduce the gauge one-form field $A^{\mathbb{A}}$ which transforms as

$$\delta A^{\mathbb{B}} = d\epsilon^{\mathbb{B}} - f_{\mathbb{C}\mathbb{D}}{}^{\mathbb{B}} A^{\mathbb{C}} \epsilon^{\mathbb{D}}.$$

The corresponding covariant derivative is

$$D_{\underset{=}{\pm}} X^\mu \equiv \partial_{\underset{=}{\pm}} X^\mu - A^{\mathbb{B}}_{\underset{=}{\pm}} k^\mu_{\mathbb{B}} \; .$$

and the invariant action becomes

$$S = \int d^2x D_{+\!\!+} X^\mu g_{\mu\nu} D_{=} X^\mu$$

Quotient 1.

Having gauged the isometry we may find a quotient defined on the space of orbits of the gauge group or alternatively construct a dual sigma model. We consider the quotient.

Starting from the gauged action for the bosonic sigma model, we integrate out the gauge fields as follows:

$$
\begin{aligned}
0 = \delta S &= \int d^2x(\delta D_{+\!\!+} X^\mu) g_{\mu\nu} D_{=} X^\nu + D_{+\!\!+} X^\mu g_{\mu\nu} \delta(D_{=} X^\nu) \\
&= -\int d^2x \left(\delta A^{\mathbb{B}}_{+\!\!+} k^\mu_{\mathbb{B}} g_{\mu\nu} D_{=} X^\nu + D_{+\!\!+} X^\mu g_{\mu\nu} \delta A^{\mathbb{B}}_{=} k^\nu_{\mathbb{B}} \right) \\
&\Rightarrow \quad k^\mu_{\mathbb{B}} g_{\mu\nu} (\partial_{=} X^\nu - A^{\mathbb{C}}_{=} k^\nu_{\mathbb{C}}) = 0 \, , \\
&\Rightarrow \quad k^\mu_{\mathbb{B}} g_{\mu\nu} \partial_{=} X^\nu - A^{\mathbb{C}}_{=} H_{\mathbb{C}\mathbb{B}} = 0 \\
&\Rightarrow \quad A^{\mathbb{C}}_{=} = H^{\mathbb{C}\mathbb{B}} k^\mu_{\mathbb{B}} g_{\mu\nu} \partial_{=} X^\nu \quad \text{where} \quad H_{\mathbb{C}\mathbb{B}} = k^\mu_{\mathbb{C}} g_{\mu\nu} k^\nu_{\mathbb{B}} \, .
\end{aligned}
$$

And similarly for $A^{\mathbb{C}}_{+\!\!+}$

Quotient 2.

We plug back this expression for the connection into the action to find

$$S \to \int d^2x \partial_{+\!\!+} X^\mu \left(g_{\mu\nu} - k_{\mu\mathbb{A}} H^{\mathbb{A}\mathbb{B}} k_{\mathbb{B}\nu} \right) \partial_= X^\nu \ .$$

The target space is the quotient $\mathcal{M}/\mathbb{G}$ where $\mathbb{G}$ is the isometry group. The red expression is the quotient metric $\tilde{g}$ which acts on orbits of $\mathbb{G}$:

$$\tilde{g}_{\mu\nu} X^\nu = \tilde{g}_{\mu\nu} (X^\nu + \alpha k^\nu)$$

Note that we may assume that $g(X + \alpha k) = g(X)$ since k generates an isometry.
In the special case of just one isometry generated by k^μ the quotient metric reads

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \ .$$

Recipe

- Start from a simple geometry on $\mathcal{M}$ with isometry group $\mathbb{G}$
- Gauge the isometries in the action
- Extremize to select a particular gauge connection
- Plug back into the action to find the reduced model on the quotient space $\mathcal{M}/\mathbb{G}$

$2d$ overview

- $\Sigma_d \to \mathcal{T}$, Symmetries dictate the geometry of $\mathcal{T}$.

$(1,1)$ susy has the same target space geometry as the bosonic model, but additional (non-maifest) supersymmetries constrain the geometry further.

B.Zumino
L.Alvares-Gaume and D. Freedman
S.J.Gates, C.Hull and M. Roček

In $2D$:

Susy	(1,1)	(2,2)	(2,2)	(4,4)	(4,4)
E=g+b	g, b	g	g, b	g	g, b
Geom	Riem.	Kähler	Biherm.	Hyperk.	Bihyperc.

Table: The geometries of sigma-models with different supersymmetries.

Hyperkähler quotient.

1. Hyperkähler geometry
2. Focus on $(4, 4)$ in $2d$, which is $\mathcal{N} = 2$ in $4d$.
3. Need representation of susy
4. Susy sigma model with $(4, 4)$
5. Killing vectors compatible with the susy
6. Susy gauging of isometries

Supersymmetry

Lorentz, Translation and Supersymmetry generators:

$$(M, P, Q, \bar{Q})$$

Schematically: $(2,2)$ in $2d$:

$$\begin{aligned} \{Q, Q\} &\sim P \\ [P, M] &\sim P \\ [Q, M] &\sim Q \\ [Q, P] &= 0 \end{aligned}$$

$$\overbrace{\{Q_+, \bar{Q}_+\}}^{Q_+\bar{Q}_+ + \bar{Q}_+ Q_+} = 2P_{\#} = 2i\partial_{\#}$$

$$\{Q_-, \bar{Q}_-\} = 2P_{=} = 2i\partial_{=}$$

Superspace

The translation generators may be represented as differential operators acting on fields over Minkowski space:

$$P_{+\!\!+} = i\partial_{+\!\!+} \ , \quad P_{=} = i\partial_{=}$$

$$P_{\substack{+\\=}}\phi = i\partial_{\substack{+\\=}}\phi(x^{+\!\!+}, x^{=})$$

Extend Minkowski space to Superspace and the fields to superfields:

$$(x^{+\!\!+}, x^{=}) \to (x^{+\!\!+}, x^{=}, \theta^{\pm}, \bar{\theta}^{\pm})$$

$$\{\theta, \theta\} = 0 \ , \quad \frac{\partial\theta}{\partial\theta} = \int d\theta\theta = 1$$

$$\phi(x^{+\!\!+}, x^{=}) \to \phi(x^{+\!\!+}, x^{=}, \theta^{\pm}, \bar{\theta}^{\pm})$$

Then the Qs may be represented as differential operators acting on superfields:

$$Q_{\pm}\phi(x, \theta, \bar{\theta}) = \left(\frac{\partial}{\partial\theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\substack{+\\=}}\right)\phi =: (\partial_{\pm} + i\bar{\theta}^{\pm}\partial_{\substack{+\\=}})\phi$$

Covariant derivatives

$$(\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm) \ , \quad \{Q, \mathbb{D}\} = 0 \ , \quad \{\bar{Q}, \mathbb{D}\} = 0 \ .$$

When acted on by these, the superfields still transform covariantly under susy. They generate another copy of the susy algebra

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = i\partial_{+\!\!+} \ , \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = i\partial_{=} \ .$$

A general superfield is not an irreducible representation of supersymmetry. To obtain irreps, use the covariant derivatives to constrain superfields:

Constrained $(2,2)$ superfields

$\bar{\mathbb{D}}_{\pm}\phi = 0$	**Chiral**
$\bar{\mathbb{D}}_{+}\chi = \mathbb{D}_{-}\chi = 0$	Twisted Chiral
$\bar{\mathbb{D}}_{+}\bar{\mathbb{D}}_{-}\Sigma_{\phi} = 0$	Complex Linear *
$\bar{\mathbb{D}}_{+}\mathbb{D}_{-}\Sigma_{\chi} = 0$	Twisted Complex Linear *
$\bar{\mathbb{D}}_{+}\ell = 0$	Left Semichiral
$\bar{\mathbb{D}}_{-}r = 0$	Right Semichiral

Components

Components by projection. Vertical bar means $\theta = \bar{\theta} = 0$.
Ex.: Minkowski components of a chiral superfield.

$$\phi(x) = \phi(x, \theta, \bar{\theta})|$$
$$\psi_{\pm}(x) = \mathbb{D}_{\pm}\phi(x, \theta, \bar{\theta})|$$
$$i\mathcal{F}(x) = \mathbb{D}_{+}\mathbb{D}_{-}\phi(x, \theta, \bar{\theta})|$$

Component supermultiplet: $(\phi, \psi_{\pm}, \mathcal{F})$.

Note that

$$i\mathbb{D}_{\pm}| = Q_{\pm}| = i\frac{\partial}{\partial\theta^{\pm}} = i\int d\theta^{\pm}$$

Reduction $(2,2) \to (1,1) \to (0,0)$

Consider the chiral fields ϕ^a and let ϕ^A denote $(\phi^a, \bar{\phi}^{\bar{a}})$. Reduce an action $(2,2) \to (1,1)$ using

$$\bar{\mathbb{D}}_{\pm} = \frac{1}{2}(D + iQ)_{\pm}$$

where $D_{\pm}$ are the $(1,1)$ covariant derivatives and $Q_{\pm}$ generate the second susys. Then the chirality constraints give

$$\bar{\mathbb{D}}_{\pm}\phi^a = \mathbb{D}_{\pm}\bar{\phi}^{\bar{a}} = 0 \iff Q_{\pm}\phi^A = J^A_B D_{\pm}\phi^B$$

with $J^A_B = diag(i\mathbf{1}, -i\mathbf{1})$. So

$$\int d^2x \mathbb{D}^2\bar{\mathbb{D}}^2 K(\phi^A)| = -\int d^2x D^2 Q^2\, K(\phi^A)|$$

Pushing in the Qs:

$$\int d^2xD^2[D_+\phi^C(K_{,CB}-J^D_C K_{,DE}\,J^E_B)D_-\phi^B]|$$

$$=\int d^2xD^2[D_+\phi^c K_{,c\bar{b}}\,D_-\bar{\phi}^{\bar{c}}+D_+\bar{\phi}^{\bar{c}}K_{,\bar{c}b}\,D_-\phi^c]|$$

A $(1,1)$ sigma model (wo b)

$$\int d^2xD^2[D_+\varphi^i g_{i\bar{k}}(\varphi,\bar{\varphi})D_-\bar{\varphi}^{\bar{k}}]$$

Chiral models 2.

Pushing in the spinorial derivatives and using the definition of the components we find

$$S = \int d^2x \Big[\partial_{+\!\!+} \phi^A g_{AB} \partial_{=} \phi^B + i\tfrac{1}{2}(\psi_+^A \nabla_= \psi_+^B + \psi_-^A \nabla_{+\!\!+} \psi_-^B) g_{AB}$$

$$-\tfrac{1}{4} R_{CDAB} \psi_+^A \psi_+^B \psi_-^C \psi_-^D \Big]$$

after eliminating the auxiliary fields $\mathcal{F}$. Now $A = (i, \bar{i})$ etc. The geometry is Kähler

$$g_{i\bar{j}} = K_{,i\bar{j}}$$

$$\Gamma_{ij}{}^{k} = g^{k\bar{s}} \partial_i g_{j\bar{s}} = K^{k\bar{s}} K_{,ij\bar{s}} \ ,$$

$$R_{i\bar{j}k\bar{s}} = g_{m\bar{j}} \partial_{\bar{s}} (\Gamma_{ik}{}^{m}) = K_{,i\bar{j}k\bar{s}} - \Gamma_{ik}{}^{m} \Gamma_{\bar{j}\bar{s}}{}^{\bar{n}} K_{,m\bar{n}}$$

Chiral models 3.

Kähler geometry is the target space geometry of $\mathcal{N} = 1$ sigma models in $4d$ and for certain $(2,2)$ sigma models in $2d$. The relation is $1-1$.

The geometry is displayed already at the $(1,1)$ level. So from now on it will be sufficient to reduce to that.

We now take a closer look at complex geometry, in particular at Kähler.

Manifold (M^{2d}, J)

Complex structure: $J \in End(TM)$ $\qquad J^2 = -\mathbf{1}$

J is real in real coordinates. $J^i{}_j J^j{}_k = -\delta^i_k$, only possible when dim $M = 2d$.

Projectors: $\quad \pi_\pm := \frac{1}{2}(\mathbf{1} \pm iJ)$

These define an involutive distribution if

$$\pi_\mp[\pi_\pm u, \pi_\pm v] = 0 \quad \Longleftrightarrow \quad \mathcal{N}(J) = 0. \text{ The Nijenhuis tensor.}$$

$$\Longleftrightarrow \quad J^k{}_m J^j{}_{[n,\,k]} - (m \leftrightarrow n) = 0.$$

This is integrability of J.

Hermitian Metric:

$$J^t g J = g \ , \quad \Longleftrightarrow \quad J^i{}_j g_{in} J^n{}_k = g_{jk}$$

$$\Rightarrow J^i{}_j g_{in} = -J^i{}_n g_{ij} \quad \Longleftrightarrow \quad J_{nj} = -J_{jn} =: \omega_{nj} \ .$$

In (canonical) complex coordinates $(z, \bar{z})$:

$$J = \begin{pmatrix} i\delta^z_z & 0 \\ 0 & -i\delta^{\bar{z}}_{\bar{z}} \end{pmatrix} \qquad g = \begin{pmatrix} 0 & g_{z\bar{z}} \\ g_{\bar{z}z} & 0 \end{pmatrix}$$

$$\omega = gJ = \tfrac{1}{2}\omega_{ij} dx^i \wedge dx^j = \omega_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}} = i g_{\bar{z}z} dz \wedge d\bar{z}$$

Kähler:

Integrable complex structure J and Hermitian metric g with further conditions.
$\exists$ Globally defined Kähler two form ω and Kähler potential $K(z,\bar{z})$:

$$d\omega = 0, \quad \nabla J = 0, \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z,\bar{z})$$

where ∇ is the Levi-Civita connection: $\nabla g = 0$.

Now $\omega = i\partial_z\partial_{\bar{z}}K(z,\bar{z})dz \wedge d\bar{z}$ so that (supressing indices) $d\omega = i\partial_z\partial_z\partial_{\bar{z}}K(z,\bar{z})dz \wedge dz \wedge d\bar{z} + c.c. = 0$ This also shows that K is only defined up to Kähler gauge transformations $K(z,\bar{z}) \to K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$.

So, in the case of the $(2,2)$ chiral superfield supersymmetric sigma model discussed above, we identify the chiral fields as canonical coordinates in terms of which the complex structure and metric are

$$J = \begin{pmatrix} i\delta^{\phi}_{\phi} & 0 \\ 0 & -i\delta^{\bar{\phi}}_{\bar{\phi}} \end{pmatrix} \qquad g = \begin{pmatrix} 0 & K_{,\phi\bar{\phi}} \\ K_{,\bar{\phi}\phi} & 0 \end{pmatrix}$$

and $K(\phi, \bar{\phi})$ is the Lagrangian for the model. The target space geometry is thus a Kähler geometry.

So we have studied a $(2,2)$ sigma model with chiral superfields and found Kähler geometry on the target space. Our goal, however, is the target space geometry of $(4,4)$ sigma model which we listed as having hyperkähler geometry. But we cannot write an action in terms of $(4,4)$ superfields. As HK geometry is a special case of Kähler geometry, we instead start from the the $(2,2)$ action and require it to have additional non manifest susy. The sigma model is

$$S = \int d^2xd^2\theta d^2\bar{\theta}K(\Phi^A)| = \int d^2xD^2[D_+\phi^i g_{i\bar{k}}(\phi,\bar{\phi})D_-\bar{\phi}^{\bar{k}}] \, ,$$

and in $(1,1)$ the ansatz for extra susy reads

$$\delta\phi^A = \epsilon^{\pm}\mathcal{J}^A_B(\phi)D_\pm\phi^B$$

We determine the matrix valued functions by two requirements

- Closure of the algebra $[\delta_1, \delta_2]\phi = -i\epsilon_1\epsilon_2\partial\phi$
- Invariance of the action $\delta S = 0$

From closure of the algebra it follows that $\mathcal{J}^2 = -1$ and $\mathcal{N}(\mathcal{J}) = 0$ (Nijenhuis).

From invariance of the action it follows that $J^t g\mathcal{J} = g$ and that $\nabla\mathcal{J} = 0$.

In our case we require two extra susys, (a total of four). The sigma model is still

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\Phi^A)| = \int d^2x D^2 [D_+ \phi^i g_{i\bar{k}}(\phi, \bar{\phi}) D_- \bar{\phi}^{\bar{k}}] \; ,$$

but the ansatz for extra susy now reads

$$\delta\phi^A = \epsilon^{\mathfrak{A}\pm} \mathcal{J}^A_{\mathfrak{A}B}(\phi) D_\pm \phi^B$$

where $\mathfrak{A} = 1, 2, 3$.

Closure of the algebra says that each $\mathcal{J}$ squares to $-\mathbf{1}$ and that its Nijenhuis tensor vanishes. Invariance of the action ensures that the metric g is hermitian with respect to each $\mathcal{J}$

$$(J^{\mathfrak{A}})^t g J^{\mathfrak{A}} = g \; ,$$

and that the Levi-Civita covariant derivative of $\mathcal{J}$ vanishes

$$\nabla J^{\mathfrak{A}} = 0 \; .$$

In fact, the complex structures satisfy a quaternion algebra.

$$J^{\mathfrak{A}} J^{\mathfrak{B}} = -\delta^{\mathfrak{A}\mathfrak{B}} + \epsilon^{\mathfrak{A}\mathfrak{B}\mathfrak{C}} J^{\mathfrak{C}} \ .$$

These conditions are the hallmark of hyperkähler geometry which we now take a closer look at

Hyperkähler geometry 1.

Special cases of Kähler geometry arise when there is more than one complex structure. A Hyperkähler manifold admits three complex structures $J^{\mathfrak{A}}$, $\mathfrak{A} = 1, 2, 3$ obeying the algebra of the quaternions:

$$J^{\mathfrak{A}} J^{\mathfrak{B}} = -\delta^{\mathfrak{A}\mathfrak{B}} + \epsilon^{\mathfrak{A}\mathfrak{B}\mathfrak{C}} J^{\mathfrak{C}}$$

This corresponds to an $SU(2)$ worth of complex structures, since any linear combination

$$a\mathcal{J}^1 + b\mathcal{J}^2 + c\mathcal{J}^3$$

is again a complex structure, provided that the real coefficients satisfy

$$a^2 + b^2 + c^2 = 1 \ .$$

The metric is Hermitean with respect to all three complex structures

$$\mathcal{J}^{t\mathfrak{A}} g \mathcal{J}^{\mathfrak{A}} = g \ , \ \ \mathfrak{A} = 1, 2, 3 \ .$$

Hyperkähler geometry 2.

The corresponding closed global two-forms are

$$\omega_{\mathfrak{A}} = g\mathcal{J}^{\mathfrak{A}} \; , \quad d\omega_{\mathfrak{A}} = 0 \; ,$$

and all complex structures are covariantly constant w.r.t. the Levi-Civita connection

$$\nabla_m (\mathcal{J}^{\mathfrak{A}})^p_q = 0$$

A complex stucture $\mathcal{J}$ takes on a simple form in holomorphic coordinates

$$\mathcal{J} = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}$$

Holomorphic coordinates z w.r.t. $\mathcal{J}^1$ will not be holomorphic for 2 and 3. Those will be related by a non-holomorphic coordinate transformation. In the z coordinates the metric is $g_{a\bar{b}} = K_{,a\bar{b}}$. For complex dimension two, the fact that the geometry is Hyperkähler results in the Monge-Ampère equation

$$\det(K_{,a\bar{b}}) = 1 \ .$$

Hyperkähler geometry 3.

- Hyperkähler is the target space geometry of $\mathcal{N} = 1$ sigma models in $6d$, $\mathcal{N} = 2$ models in $4d$ and $(4, 4)$ sigma models in $2d$.
- The curvature tensor governs the transformation $T_p(\mathcal{M}) \to T_p(\mathcal{M})$ that results from parallel transporting vectors along closed loops. When these transformations are globally defined, they form the Holonomy group of $\mathcal{M}$. A manifold may be characterised by its Holonomy group. The holonomy group of a $\dim_C = m$ Kähler manifold is contained in $U(m)$, for a Calabi-Yau manifold it is contained in $SU(m)$ and for a Hyperkähler manifold of $\dim_C = 2n$ in the symplectic group $Sp(n)$.

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