Supersymmetry, complex geometry and the hyperkähler quotien II

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Gauging Isometries 2.

For $(1,1)$ sigma model with isometries the gauging is parallel to that of the bosonic model.

$$S = \int d^2x D_+ D_- \Big(D_+\phi^\mu(x,\theta) E_{\mu\nu}(\phi) D_-\phi^\nu(x,\theta) \Big)_|$$

$$\to \int d^2x D_+ D_- \Big(\mathcal{D}_+\phi^\mu E_{\mu\nu}(\phi) \mathcal{D}_-\phi^\nu \Big)_|$$

where

$$\mathcal{D}_\pm\phi^\mu = D_\pm\phi^\mu - A_\pm^{\mathbb{B}} k_{\mathbb{B}}^\mu(\phi)$$

Quotient for $(1,1)$ models: The story is again almost identical to the bosonic case. We integrate out the gauge connections

$$0 = \delta S \;\; = \;\; \int d^2 x D_+ D_- \Big(\delta\mathcal{D}_+\phi^\mu g_{\mu\nu}(\phi)\mathcal{D}_-\phi^\nu + \mathcal{D}_+\phi^\mu g_{\mu\nu}(\phi)\delta\mathcal{D}_-\phi^\nu \Big)_|$$

where

$$\mathcal{D}_\pm\phi^\mu = D_\pm\phi^\mu - A_\pm^{\mathbb{B}} k_{\mathbb{B}}^\mu(\phi) \; .$$

This gives the quotient model

$$S = \int d^2 x D_+ D_- \big(D_+\phi^\mu \big(g_{\mu\nu} - k_{\mu\mathbb{A}} H^{\mathbb{A}\mathbb{B}} k_{\mathbb{B}\nu} \big) D_-\phi^\nu \big) \; .$$

The $(1,1)$ supersymmetry is preserved through the use of $(1,1)$ superfields. For extended supersymmetry the story gets more complicated.

Symplectic quotient: The gauged $N = 2$ nonlinear sigma model

An $N = 2$ sigma model has the action

$$S = \int d^3x \mathbb{D}^2 \bar{\mathbb{D}}^2 K(\Phi, \bar{\Phi})$$

Isometries X will in general only leave the potential K invariant up to a Kähler gauge transformation

$$XK = \nu^X(\Phi) + \bar{\nu}^X(\bar{\Phi}) \ .$$

This is sufficient since the holomorphic function ν^X is annihilated by the measure $\mathbb{D}^2 \bar{\mathbb{D}}^2$.
The Kähler gauge transformation complicates the gauging procedure and we will need some new concepts.

A holomorphic isometry X on a Kähler manifold satisfies

$$\mathcal{L}_X J = 0 \,, \quad \mathcal{L}_X \omega = 0$$

where J is the complex structure and ω the Kähler two-form. It follows that

$$\imath_X \omega =: d\mu^X$$

In holomorphic coordinates $(\phi^q, \bar{\phi}^{\bar{q}})$ where $X = X^{\mathbb{A}}(k_{\mathbb{A}} + \bar{k}_{\mathbb{A}})$, this reads

$$\omega_{p\bar{q}} X^{\mathbb{A}} k^p_{\mathbb{A}} \equiv -i K_{\bar{q}p} X^{\mathbb{A}} k^p_{\mathbb{A}} = \frac{\partial \mu^X}{\partial \bar{\phi}^{\bar{q}}} =: \mu^X_{\bar{q}} \,.$$

μ^X Killing potential, (Hamiltonian function),
$\mu : \mathcal{M} \to \mathfrak{g}^*$ moment map.

So, from the Kähler gauge transformation we now have for a holomorphic isometry

$$\tfrac{1}{2}\,(1 - iJ)\,XK = X^{\mathbb{A}} k^{p}_{\mathbb{A}}(\phi)\partial_p K = -i\mu^X + \nu^X \ .$$

This will be part of the story when checking invariance once we have defined the gauged action.
No minimal coupling. Instead the antichiral fields couple to a real superfield V with the vector potential as one of its components. In the case of isotropy, $k^i_{\mathbb{A}}(\phi) = (T_{\mathbb{A}})^i_j \phi^j$,

$$\bar{\phi}^i \to \tilde{\bar{\phi}}^i := (e^V)_j{}^i \bar{\phi}^j = \left(e^{\mathcal{L}_{iV}}\bar{\phi}\right)^i \ , \qquad (V)_j{}^i = V^{\mathbb{A}}(T_{\mathbb{A}})_j{}^i \ .$$

This is an isometry transformation

$$\bar{\phi}^i \to \bar{\phi}^j (e^{-i\bar{\Lambda}})_j{}^i \ , \qquad (\bar{\Lambda})_j{}^i = \bar{\Lambda}^{\mathbb{A}}(T_{\mathbb{A}})_j{}^i \ .$$

with (chiral) parameter $\bar{\Lambda} = iV$.

if K is invariant (no Kähler gauge tfs) under the un-gauged isometries the coupling is thus

$$K(\phi, \bar{\phi}) \to K(\phi, \tilde{\phi}) \ .$$

If the invariance is up to KGTs we are still not OK, since

$$\delta S = \int X^{\mathbb{A}} \bar{\nu}_{\mathbb{A}}(\bar{\phi})$$

with X^A constant, now reads

$$\delta S = \int \bar{\Lambda}^{\mathbb{A}}(\Phi, \bar{\Phi}) \bar{\nu}_{\mathbb{A}}(\bar{\phi})$$

and will not vanish.

The remedy is to introduce auxiliary (anti)chiral superfields ζ and $\bar{\zeta}$. The extended potential

$$\tilde{K} := K(\phi, \bar{\phi}) - \zeta - \bar{\zeta}\,.$$

is now *IN*variant under isometries generated by

$$k'_{\mathbb{A}} = k^{p}_{\mathbb{A}} \partial_p + \nu_{\mathbb{A}} \partial_\zeta$$
$$\bar{k}'_{\mathbb{A}} = k^{\bar{p}}_{\mathbb{A}} \partial_{\bar{p}} + \bar{\nu}_{\mathbb{A}} \partial_{\bar{\zeta}}$$

Gauging is now straight forward

$$\tilde{K}(\phi, \zeta, \bar{\phi}, \bar{\zeta}) \to \tilde{K}(\phi, \zeta, \tilde{\bar{\phi}}, \tilde{\bar{\zeta}}) = K(\phi, \tilde{\bar{\phi}}) - \zeta - \bar{\zeta} + \frac{e^{\mathcal{L}_{iV}} - 1}{\mathcal{L}_{iV}} \bar{\nu}^{V}$$

Expanding $\tilde{\bar{\phi}}$, rewriting the expression in a manifestly hermitian form and making use of the relations between KGTs and moment map yields the final form.

Then the $N = 2$ gauged sigma model reads

$$\hat{K}(\Phi, \bar{\Phi}, V) = K(\Phi, \bar{\Phi}) - \int_0^1 dte^{tY} \mu^V$$

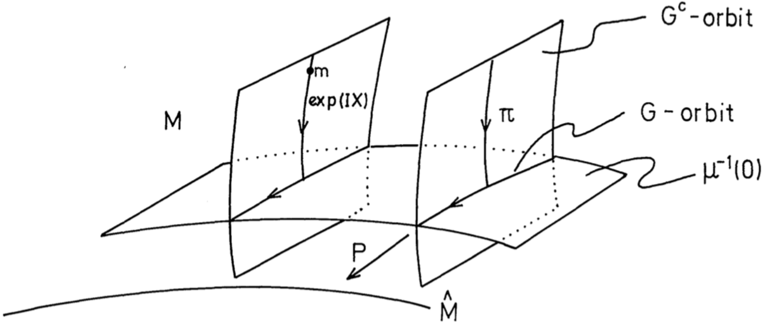
where $Y := V\dot{J}k$ and

$$\mu^V = V^{\mathbb{A}} \mu_{\mathbb{A}} \; .$$

Eliminating $V_{\mathbb{A}}$ yields

$$e^Y \mu_{\mathbb{A}} = 0 \; .$$

Solving this for $V_{\mathbb{A}}$ and plugging the solution back produces the Kähler quotient model, as the following example illustrates:


G^c-orbit
m
exp(IX)
M
π
G-orbit
μ^-1(0)
P
M̂

Two chiral fields Φ^p, $p = 1, 2$. Potential

$$K(\Phi^p, \bar{\Phi}^{\bar{p}}) = \Phi^p \bar{\Phi}^{\bar{p}} \ ,$$

i.e., the Kähler potential for C^2. Then

$$X\Phi^p = i\xi\Phi^p \ , \quad X\bar{\Phi}^{\bar{p}} = -i\xi\bar{\Phi}^{\bar{p}}$$

generate a $U(1)$ isometry. The moment map is

$$\mu^\xi = \xi(K - c)$$

with c a constant. The gauged potential becomes

$$\hat{K}(\Phi, \bar{\Phi}, V) = \Phi^p \bar{\Phi}^{\bar{p}} e^V - cV$$

Solving the V-equation gives

$$V = -ln\left(\frac{\Phi^p\bar{\Phi}^{\bar{p}}}{c}\right)$$

Plugging this back we find

$$\hat{K}(\Phi, \bar{\Phi}, V(\Phi^p\bar{\Phi}^{\bar{p}})) = c\, ln(1 + \varphi\bar{\varphi}) + f(\Phi) + \bar{f}(\bar{\Phi})$$

where

$$\varphi = \frac{\Phi^1}{\Phi^2}$$

and we recognise the potential for the Fubini Study metric for $\mathbb{CP}1$, up to the Kähler gauge transformation $f + \bar{f}$.

The gauged $N = 4$ nonlinear sigma model.

Hyperkähler geometry is a special case of Kähler geometry with three complex structures $J^{\mathfrak{A}}$, $\mathfrak{A} = 1, 2, 3$ satisfying a quaternion algebra:

$$J^{\mathfrak{A}} J^{\mathfrak{B}} = -\delta^{\mathfrak{A}\mathfrak{B}} + \epsilon^{\mathfrak{A}\mathfrak{B}\mathfrak{C}} J^{\mathfrak{C}}$$

The $N = 2$ model describing Kähler geometry describes hyperkähler geometry if there are supersymmetries corresponding to all these complex structures, In terms of $N = 1$ fields

$$\delta\phi^p = J_q^{\mathfrak{A}p} \epsilon_{\mathfrak{A}}^{\alpha} D_\alpha \phi^q$$

or in $N = 2$

$$\delta\Phi^p = \bar{\mathbb{D}}^2(\bar{\epsilon}\Omega^p)$$

with ϵ a constant superfield parameter obeying certain constraints and the non-manifest complex structures related to the function Ω.

The $N = 4$ Lie algebra valued gauge multiplet now becomes

$$(V, S, \bar{S})$$

where V is a real $N = 2$ scalar superfield and S is a complex chiral. Likewise, we now have a multiplet of moment maps

$$(\mu, \mu_{\pm})$$

The isometries we use are tri holomorphic, i.e., holomorphic wrt all three complex structures.

Let

$$\mu_+^S = S^{\mathbb{A}} \mu_{\mathbb{A}}^+ \ , \ \ \mu_-^S = \bar{S}^{\mathbb{A}} \mu_{\mathbb{A}}^-$$

then

$$S = \int d^3x \left[\mathbb{D}^2 \bar{\mathbb{D}}^2 \hat{K}(\Phi, \bar{\Phi}, V) + \tfrac{1}{2} \mathbb{D}^2 \mu_+^S + \tfrac{1}{2} \bar{\mathbb{D}}^2 \mu_-^{\bar{S}} \right]$$

has $N = 4$ susy and is invariant under the gauged isometry transformations

$$\delta \Phi^p = \bar{\mathbb{D}}^2 (\bar{\epsilon} e^{\bar{Z}} \Omega^p) \quad , \quad Z := i V^{\mathbb{A}} k_{\mathbb{A}}^p \partial_p$$

We find the quotient action by eliminating the $N = 4$ gauge fields $(V, S, \bar{S})$.
All global issues are under control.